Coupling conditions for traffic flow networks

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November 3, 2005

Abstract

We consider coupling conditions for the "Aw–Rascle" traffic flow model at an arbitrary road intersection. In contrast with coupling conditions previously introduced in [12] and [9], we require only that all the moments of the "Aw–Rascle" system are conserved **and** the total flux at the junction is maximized. This nonlinear optimization problem is solved explicitly. We show how the two simple cases of merging and diverging junctions can be extended to more complex junctions, like roundabouts. Finally, we present some numerical results.

AMS subject classifications: 35Lxx, 35L6

1 Introduction

Traffic flow models have been under investigation for a long time. We are particularly interested in macroscopic traffic flow models based on hyperbolic conservation laws. Models of this type have been considered for example in [16, 7, 17, 2, 1, 10]. In the following, we focus on the "Aw–Rascle" (AR) model. This (class of) "second–order" model(s) consists of a nonlinear, coupled system of conservation laws, introduced in [2] and independently in [18]. Those models describe the behavior of traffic density and velocity where different cars can have a different response to local traffic situations, e.g., the model distinguishes trucks and cars. Recently, first extensions of theses models to a traffic network have been proposed [9, 12]. The crucial point is the modelling of coupling conditions at junctions. Typically, one has to introduce further assumptions to show that the problem is well-defined and admits a unique solution, see also the discussion in the scalar case [5, 13, 11]. In this paper we propose new coupling conditions for the AR-system. In contrast with [9], those conditions conserve all moments of the system and in contrast with [12] the derived conditions maximize the flux at the junctions without any further constraint. Furthermore, we present a numerical algorithm to solve the problem and to construct the intermediate states of the homogenized solution.

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2 Preliminary discussion

We first give a brief summary of the properties of the AR–model and advise the reader to consult [2, 12] for more details.

A road network is modelled as finite, directed graph $(\mathcal{K}, \mathcal{N})$ wherein each arc $k = 1, \ldots, \mathcal{K}$ corresponds to a road and each vertex $n \in \mathcal{N}$ to a junction. For a fixed junction n the set δ_n^- contains all the indices k of incoming roads to n. Similarly, δ_n^+ denotes the indices j of outgoing roads. We skip the subindex n whenever the situation is clear. Each road k is modelled by an interval $I_k := [a_k, b_k]$ where we allow either $a_k = -\infty$ or $b_k = +\infty$ for arcs k incoming or outgoing to the whole network. We require the AR–equations (1) to hold on each arc $k \in \mathcal{K}$ of the network:

$$\partial_t \rho_k + \partial_x(\rho_k v_k) = 0 \tag{1a}$$

$$\partial_t(\rho_k w_k) + \partial_x(\rho_k v_k w_k) = 0 \tag{1b}$$

$$w_k = v_k + p_k(\rho_k) \tag{1c}$$

where, for each k, $\rho \mapsto p_k(\rho)$ is a known function ("traffic pressure") with the following properties

$$\forall \rho, \rho p_k''(\rho)) + 2p_k'(\rho) > 0 \text{ and e.g. } p_k(\rho) \sim \rho^\gamma \text{ at } \rho = 0, \tag{2}$$

and where $\gamma > 0$. ρ_k and v_k respectively, describe the density and velocity of traffic on road k.

The conservative form of (1) is

$$\partial_t \begin{pmatrix} \rho_k \\ y_k \end{pmatrix} + \partial_x \begin{pmatrix} y_k - \rho_k p_k(\rho_k) \\ (y_k - \rho_k p_k(\rho_k)) y_k / \rho_k \end{pmatrix} = 0, \tag{3a}$$

where $y_k := \rho_k w_k = \rho(v_k + p_k(\rho_k))$.

We now recall some basic facts on the solution of the Riemann Problem for (1), i.e. to the initial value problem with constant data for $\pm x > 0$.

The system is strictly hyperbolic if $\rho_k > 0$. The eigenvalues are

$$\lambda_{1,k}(U) = v - \rho p_k'(\rho) \text{ and } \lambda_{2,k}(U) = v \tag{4}$$

The first characteristic field is *genuinely nonlinear*. The second is *linearly degenerate* and therefore associated with a *contact discontinuity*. Moreover, the *1–shock* and *1–rarefaction* curves *coincide*, see [6, 2]. We recall that of course they are associated with braking and acceleration waves, respectively. For each fixed k, the Riemann invariants are

$$\mathbf{w}_k(U) = v + p_k(\rho) \text{ and } \mathbf{v}_k(U) = v. \tag{5}$$

We refer to [12, 9, 13] for a derivation of the necessary conditions at the junction, i.e. the coupling conditions. First, we define weak solutions of the network problem in the following sense: for any set of smooth functions $\{\phi_k\}_{k \in \mathcal{K}} : [0, +\infty[\times I_k \to \mathbb{R}^2$ having compact support in $I_k = [a_k, b_k]$, which are smooth across a junction n, i.e. such that

$$\phi_k(b_k) = \phi_j(a_j) \; \forall k \in \delta_n^- \text{ and } \forall j \in \delta_n^+. \tag{6}$$

Then a set of functions $\{U_k = (\rho_k, \rho_k v_k)\}_{k\in\mathcal{K}}$ is called a weak solution of (1) if and only if

$$\sum_{k=1}^{\mathcal{K}} \int_0^\infty \int_{a_k}^{b_k} \begin{pmatrix} \rho_k \\ \rho_k w_k \end{pmatrix} \cdot \partial_t \phi_k + \begin{pmatrix} \rho_k v_k \\ \rho_k v_k w_k \end{pmatrix} \cdot \partial_x \phi_k dx dt$$
$$+ \int_{a_k}^{b_k} \begin{pmatrix} \rho_{k,0} \\ \rho_{k,0} w_{k,0} \end{pmatrix} \cdot \phi_k(x,0) dx = 0 \tag{7a}$$

holds. Herein, $U_{k,0}(x) = \big(\rho_{k,0}(x), (\rho_{k,0} v_{k,0})(x)\big)$ is the initial data. Furthermore, the set of functions U_k satisfies for all k the relation

$$w_k(x,t) = v_k(x,t) + p_k^\dagger(\rho_k(x,t)) \tag{8}$$

where the function $p_k^\dagger(\cdot)$ is initially unknown. On an outgoing road its explicit form depends on the mixture of the cars. On any incoming roads $k \in \delta^-$ it holds $p_k^\dagger \equiv p_k$.

From now on, we consider a **single** junction. Then, from (7), (8) we derive the Rankine–Hugoniot condition for piecewise smooth solutions

$$\sum_{k\in\delta^-} (\rho_k v_k)(b_k^-, t) = \sum_{j\in\delta^+} (\rho_j v_j)(a_j^+, t) \tag{9a}$$
$$\sum_{k\in\delta^-} (\rho_k v_k w_k)(b_k^-, t) = \sum_{j\in\delta^+} (\rho_j v_j w_j)(a_j^+, t) \tag{9b}$$

These properties, respectively, correspond to conservation of mass and(pseudo)-“momentum”. Note that in [9] the pseudo-"’momentum"’ is not conserved and the proposed solution is **not** a weak solution in the above sense.

In the remaining part we consider the case of initial data constant on each road:

$$(\rho_{k,0}, \rho_{k,0} v_{k,0}) = U_{k,0} = \text{const}_k \tag{10}$$

We construct a weak solution in the above sense by considering the following list of (half-)Riemann problems. To be more precise, for each $k \in \delta^- \cup \delta^+$, we look for solutions to

$$\partial_t \begin{pmatrix} \rho_k \\ \rho_k w_k \end{pmatrix} + \partial_x \begin{pmatrix} \rho_k v_k \\ \rho_k v_k w_k \end{pmatrix} = 0,\ U_k(x,0) = \begin{pmatrix} U_k^- & x < x_0 \\ U_k^+ & x > x_0 \end{pmatrix} \tag{11}$$

Hence, for $k \in \delta^-$ the left datum U_k^- is given by the initial condition (and $x_0 = b_k$). For $j \in \delta^+$ the right initial datum U_j^+ is given (and $x_0 = a_j$). For each k the remaining **unknown** state U_k^+ (resp. U_k^-) has to be determined in such way that the coupling conditions (9a) and (9b) are satisfied. Then we solve each of the problems (11) and we obtain a weak solution to (7) restricting the solution $U_k(x,t)$ of (11) to $x < x_0$ for $k \in \delta^-$ and to $x > x_0$ for $k \in \delta^+$.

Summarizing, **depending** on the road, we only know a part of the initial data for (11) :

$$U_k^- = U_{k,0}\,,\ x_0 = b_k,\ \forall k \in \delta^- \text{ and } U_j^+ = U_{j,0}\,,\ x_0 = a_j,\ \forall j \in \delta^+. \tag{12}$$

First, we denote by α_{jk} the percentage of cars on road k willing to go (and actually going, see below) to road j. The corresponding matrix $A :=$

$(\alpha_{jk})_{j\in\delta^+,k\in\delta^-}$ is assumed to be **known**, see [5, 9, 12]. By definition we have

$$\sum_{j\in\delta^+} \alpha_{jk} = 1 \; \forall k \in \delta^-. \tag{13}$$

Next, let

$$q_k(t) := \rho_k v_k(b_k-, t), \; q_j(t) := \rho_j v_j(a_j+, t)$$

denote the (initially unknown) total flux on the incoming road k (resp. on the outgoing road j). Furthermore, let us introduce the (initially unknown) flux q_{jk} of cars actually going from road k to road j and let $\beta_{jk} := q_{jk}/q_j$, which is also initially unknown. Then, by the above definitions,

$$\alpha_{jk} = \frac{q_{jk}}{q_k} \text{ and } \sum_{k\in\delta^-} \beta_{jk} = 1.$$

As a final preparation, we describe the construction of the demand and supply functions for a given level curve of $w := w_k$ $\{\mathbf{w}(U) = c\}$, $c \geq 0$. Recall, $\mathbf{w}(U) = v + p(\rho)$ and its level curve is a concave function in the $(\rho, \rho v)$-plane with a unique maximum. As in the case of first-order models, e.g. [15, 9], in the $(\rho, \rho v)$ plane the demand function $d(\rho; \mathbf{w}, c)$ is an extension of the **non–decreasing** part of this level curve $\{\mathbf{w}(U) = c\}$ for $\rho \geq 0$ and the supply function $s_j(\rho; \mathbf{w}, c)$ is an extension of the **non–increasing** part of this curve $\{\mathbf{w}(U) = c\}$ and $\rho \geq 0$. We denote by $d_k(\rho; \mathbf{w}, c)$ the demand on an *incoming* road k and by s_j the supply on an *outgoing* road j.

3 Solving the problem at a junction

We first recall the following results of [12].

Proposition 3.1. *Consider an incoming (resp. outgoing) road i, an initial datum $U^- := U_{k,0}$ (resp. $U^+ := U_{k,0}$) and an arbitrary flux $q_0 \geq 0$. Let $\mathbf{w}(U) := v + p_k(\rho)$ and $c := w(U_{k,0})$. Assume*

$$q_0 \leq d(\rho_{k,0}; \mathbf{w}, c)\;, \; \text{(resp. } q_0 \leq s(\rho_{k,0}; \mathbf{w}, c)).$$

Then there exists a unique state U^+ (resp. U^-), such that the corresponding Riemann problem (11) admits a solution $\rho^+v^+ = q_0$ (resp. $\rho^-v^- = q_0$) and *either all the waves have negative (resp. positive) speed, or the solution is a constant on the corresponding road.*

An example is given in Figure 1 for incoming and in Figure 2 for outgoing roads, respectively.

Next, we describe how to construct a solution for a single junction and constant initial data $U_{k,0}$. We assume that the weak solution (7) satisfies the already imposed condition,

$$q_j \equiv (\rho_j v_j)(a_j+, t) = \sum_{k\in\delta^-} q_{jk} = \sum_{k\in\delta^-} \alpha_{jk}(\rho_k v_k)(b_k-, t) \; \forall j \in \delta^+. \tag{15}$$

As already noted in [12, 9] the condition (15) is not sufficient to construct a unique solution $\{U_k\}_k$. In [12] we introduced a further assumption on the distribution of the cars. Here, and in contrast with [12] and with [9] we present a *new*

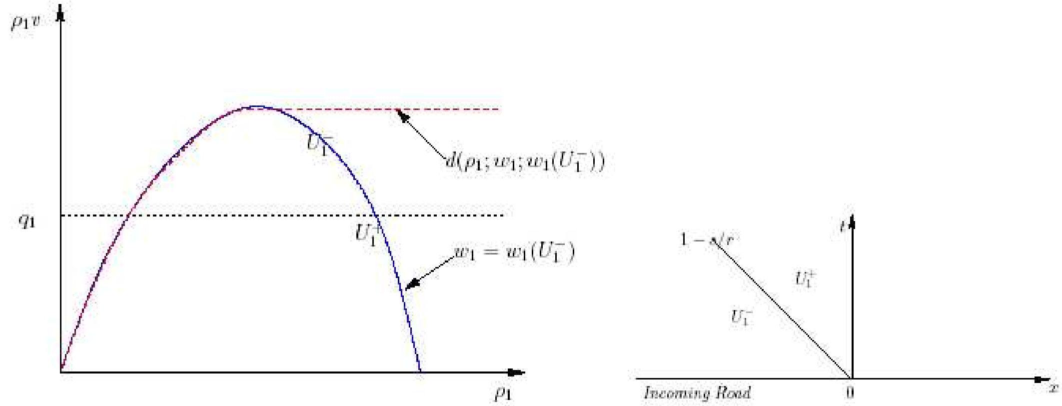


Figure 1: (Half-)Riemann Problem on an incoming road.

approach to solve the problem: We prove that solving the maximization of the total incoming flux $\sum_{k\in\delta^-}\rho_k v_k(b_k-,t)$ is sufficient to obtain a **unique** solution. Furthermore, this solution will satisfy assumptions (15) for a given matrix A with the property (13). In contrast with [9], **no** additional assumptions on A is needed. However, we maximize the total flux on a smaller set of admissible states. On the other hand, in contrast with [12], **no** additional assumption on β_{ji} (i.e. the mixture of the cars) has to be imposed.

At this point, using Theorem 7.1 in [12] we want to define admissible solutions for a general junction.

Definition 3.1. *Consider a junction with m incoming and n outgoing roads, with constant initial data $U_{k,0} = (\rho_{k,0}, \rho_{k,0}v_{k,0})$ under assumptions (13).*

We say that the family $\{U_k(x,t)\}_{k\in\delta^-\cup\delta^+}$ is an admissible solution of the Riemann problems (11) and (12) if and only if it satisfies:

(1) For all $k \in \delta^- \cup \delta^+$, $U_k(x,t)$ is a weak entropy solution of the network problem (7),(8), where $p_k^\dagger \equiv p_k, \forall k \in \delta^-$.

On the outgoing roads $j \in \delta^+$, the solution $U_j(x,t)$ is constructed as in [12]: in the triangle $\{(x,t); a_j < x < a_j + tv_{j,0}\}$, U_j is the homogenized solution defined below with $p_j^\dagger \equiv p_j^*$, whereas for $x > a_j + tv_{j,0}$, $p_j^\dagger \equiv p_j$.

(2) The flux distribution satisfies (15).

(3) The sum of the incoming fluxes $\sum_{k\in\delta^-}\rho_k v_k(b_k-,t)$ is maximal subject to (1) and (2).

Remark 3.1. *We recall that in [9] the maximization problem involves different cost functions and mostly a larger set of admissible states, since relation (9b) is not imposed. In contrast, in [12] no such maximization criterion is imposed, but the proportions between all the incoming fluxes are fixed.*

Next, we describe the homogenization to define p_j^*. For a motivation and a detailed discussion of the homogenization, see [3] and Section 6 in [12]. Recall

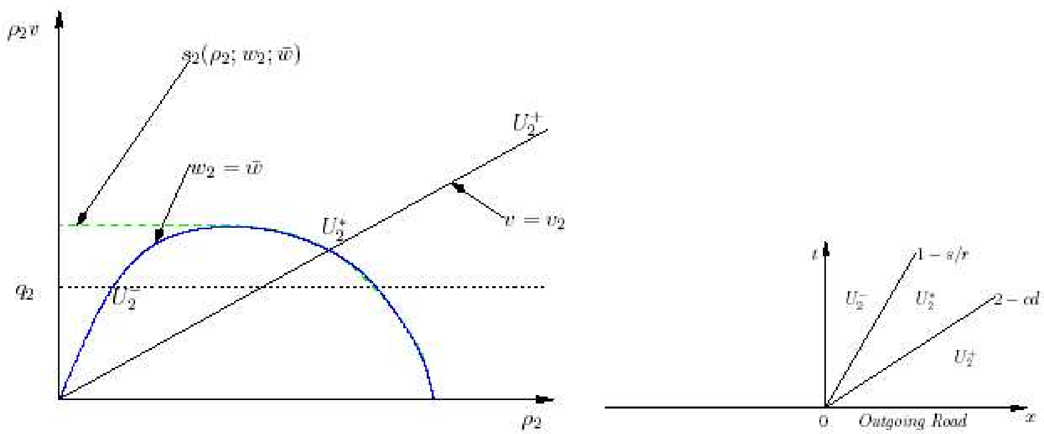


Figure 2: (Half-)Riemann Problem on an outgoing road.

that for each $k \in \delta^-$, $p_k^\dagger \equiv p_k$ and $\mathbf{w}_k(U) = v + p_k(\rho)$ are well-defined. First, we define the (initially unknown) homogenized value for each outgoing road $j \in \delta^+$

$$\bar{w}_j := \sum_{k \in \delta^-} \beta_{jk} \mathbf{w}_k(U_{k,0}). \tag{16}$$

Then, for each $j \in \delta^+$, $p_j^*(\cdot)$ is defined as in [12]. Namely, we first define the function

$$P_j(\tau) := p_j(1/\tau), \tag{17}$$

where $\tau = \frac{1}{\rho}$ is the specific volume, see [1, 3].

Now, we consider the function:

$$v \longmapsto \tau := \sum_{k \in \delta^-} \frac{q_{jk}}{q_k} P_j^{-1}(\mathbf{w}_k(U_{k,0}) - v) = \sum_{k \in \delta^-} \beta_{jk} P_j^{-1}(\mathbf{w}_k(U_{k,0}) - v). \tag{18}$$

Then, we *choose* to define a new invertible function P_j^* by rewriting (18) under the form

$$\tau := \left(P_j^*\right)^{-1} (\bar{w}_j - v), \tag{19}$$

which we **only** use with the particular value $\bar{w}_j$ defined by (16), see [3] for more details.

Finally, we set

$$p_j^\dagger(\rho) := p_j^*(\rho) := P_j^*(1/\rho), \tag{20a}$$

$$\mathbf{w}_j^\dagger(U) := v + p_j^\dagger(\rho). \tag{20b}$$

This construction is perfectly well–defined once the proportions $\beta_{jk} = q_{jk}/q_k$ are known. In [12] we have assumed that we knew these proportions a priori, see also Remark 3.1. Here, in contrast, we show that the proportions β_{jk} can be determined by solving a maximization problem stated below. Unfortunately, so far, the problem is only tractable for particular types of junctions. With all the previous remarks in mind, we conclude: there exists a unique solution $\{U_k\}_k$ in the sense of Definition 3.1 if the maximization problem (21) has a unique solution.

$$\max \sum_{k\in\delta^-\cup\delta^+} q_k \text{ subject to} \tag{21a}$$

$$\forall k \in \delta^-,\ 0 \le q_k \le d_k(\rho_{k,0}, \mathbf{w}_k, \mathbf{w}_k(U_{k,0})), \tag{21b}$$

$$\forall j \in \delta^+,\ 0 \le q_j \le s_j\left(\rho_{j,0}, \mathbf{w}_j^\dagger, w_j^*)\right), \tag{21c}$$

$$\forall k \in \delta^-, \forall j \in \delta^+,\ \beta_{jk} = \frac{q_{jk}}{q_k}, \tag{21d}$$

$$\forall k \in \delta^-, \forall j \in \delta^+,\ \alpha_{jk} = \frac{q_{jk}}{q_k}, \tag{21e}$$

$$\forall j \in \delta^+,\ \sum_{k\in\delta^-} \beta_{jk} = 1, \tag{21f}$$

$$\forall j \in \delta^+,\ q_k = \sum_{k\in\delta^-} \alpha_{jk} q_k, \tag{21g}$$

$$\forall j \in \delta^+,\ \sum_{k\in\delta^-} q_{jk} = q_j, \tag{21h}$$

$$\forall k \in \delta^-,\ \sum_{j\in\delta^+} q_{jk} = q_k,, \tag{21i}$$

$$\forall k \in \delta^-, \forall j \in \delta^+,\ 0 \le \beta_{jk} \le 1. \tag{21j}$$

with $\mathbf{w}_j^\dagger \equiv v + p_j^\dagger(\rho)$ and $w_j^* \equiv \sum_{k\in\delta^-} \beta_{jk}\mathbf{w}_k(U_{k,0}$

Remark 3.2. *The functions $\mathbf{w}_k$ for $k \in \delta^-$ and the values $\mathbf{w}_k(U_{k,0})$ and α_{jk} are initially known. As already noted, β_{jk} is initially unknown and depends on the solution $\{U_k\}_k$, as well as the function $\mathbf{w}_j^\dagger$ and the fluxes q_{jk} and q_j. In particular the maximization problem contains the implicit constraints (21c), for all $j \in \delta^+$.*

In (21) some equations are redundant and a more compact equivalent reformulation is:

$$\max \sum_{j\in\delta^+} q_j \text{ subject to} \tag{22a}$$

$$\forall k \in \delta^-,\ 0 \le q_k \le d_k(\rho_{k,0}; \mathbf{w}_k, \mathbf{w}_k(U_{k,0})), \tag{22b}$$

$$\forall j \in \delta^+,\ 0 \le q_j \le s_j(\rho_{j,0}, \mathbf{w}_j^\dagger, w_j^*), \tag{22c}$$

$$\forall k \in \delta^-, \forall j \in \delta^+,\ \beta_{jk} q_j = \alpha_{jk} q_k, \tag{22d}$$

$$\forall j \in \delta^+,\ \sum_{k\in\delta^-} \beta_{jk} = 1, \tag{22e}$$

$$\forall k \in \delta^-, \forall j \in \delta^+,\ 0 \le \beta_{jk} \le 1. \tag{22f}$$

We now move to the first type of junctions considered here (a merge).

4 Two incoming roads and one outgoing road

We write $i = 1, 2$ for incoming roads, $j = 3$ for the outgoing road and give simplifications for (22) in this case. Furthermore, let $\beta_1 := \beta_{31}$ and $\beta_2 := \beta_{32}$

and $d_1 := d(\rho_{1,0}; \mathbf{w}_1, \mathbf{w}_1(U_{1,0}))$ and $d_2 := d(\rho_{2,0}; \mathbf{w}_2, \mathbf{w}_2(U_{2,0}))$. By assumption (13), $\alpha_{31} = \alpha_{32} = 1$.

The crucial point in solving (22) is to determine the supply s_j. We briefly describe the homogenization leading to s_j, before describing the solution.
Let $U := (\rho, \rho v)$. In this particular case, the homogenization process described in Section 3 can be rewritten as follows.
First, on each incoming road $i = 1, 2$, the curve $\{\mathbf{w}_i(U) = \mathbf{w}_i(U_{i,0})\}$ becomes in Lagrangian coordinates:

$$\{U(\tau, v); v + P_i(\tau) = \mathbf{w}_i(U_{i,0})\}.$$

Now, on the outgoing road 3, the general equations (16) to (20) become

$$\bar{w} = \beta_1 \mathbf{w}_1(U_{1,0}) + (1 - \beta_1)\mathbf{w}_2(U_{2,0}) \tag{23}$$

and

$$\tau = \beta_1 P_1^{-1}(\mathbf{w}_1(U_{1,0}) - v) + (1 - \beta_1) P_2^{-1}(\mathbf{w}_2(U_{2,0}) - v), \tag{24}$$

where β_1 is still unknown.

As a prototype, we treat the case where $p_k(\rho) = \rho^\gamma$ (or $P_k(\tau) = 1/\tau^\gamma$) for $k = 1, 2, 3$, with $\gamma = 1$. Then, (24) becomes

$$\tau_3 = \frac{\beta_1}{w_1 - v} + \frac{(1 - \beta_1)}{w_2 - v}, \tag{25}$$

where w_i is the ***constant*** $w_i := \mathbf{w}_i(U_{i,0})$, $i = 1, 2$.
This homogenized relation (25) implies

$$\rho_3 v = \frac{(w_2 - v)(w_1 - v)v}{\beta_1(w_2 - w_1) + w_1 - v}. \tag{26}$$

Combining (26), (25), problem (22) is equivalent to solving the following maximization problem

$$\max q_3 \text{ subject to} \tag{27a}$$
$$0 \le q_3 \le \frac{d_1}{\beta_1}; \tag{27b}$$
$$0 \le q_3 \le \frac{d_2}{(1 - \beta_1)}; \tag{27c}$$
$$0 \le q_3 \le s_3(U_{3,0}, \mathbf{w}_3^\dagger, \mathbf{w}_3^*); \tag{27d}$$
$$0 \le \beta_1 \le 1. \tag{27e}$$

We set $v_3 := v_{3,0}$. Then, for each given β_1, we denote by v_c the velocity corresponding to the maximal flux on the outgoing road, according to the supply i.e, v_c is obtained by solving $\frac{d(\rho_3 v)}{dv} = 0$ for any *fixed* β_1. The supply $s_3(U_{3,0}, \mathbf{w}_3^\dagger, \mathbf{w}_3^*)$ is then:

$$s_3 = \begin{cases} \frac{(w_2 - v_3)(w_1 - v_3)v_3}{\beta_1(w_2 - w_1) + w_1 - v_3} & if \quad v_3 \le v_c; \\ \\ \frac{(w_2 - v_c)(w_1 - v_c)v_c}{\beta_1(w_2 - w_1) + w_1 - v_c} & if \quad v_3 > v_c. \end{cases} \tag{28}$$

For any fixed v_3, we note that the function $\beta_1 \longmapsto s_3(v_3, \beta_1)$ is non–decreasing if $w_1 > w_2$, non–increasing if $w_1 < w_2$ and constant if $w_1 = w_2$, in which case the homogenization problem is trivial.

Now, we can solve the maximization Problem (27). Since the problem is symmetric with respect to the incoming roads, it suffices to consider the case $w_1 \geq w_2$, see below.

4.1 The case $w_1 > w_2$

The optimal solution of the Problem (27) is reached only in one of the following cases:

4.1.1 Case 1: $q_3 < s_3(v_3, \beta_1)$

Subcase 1.1 In this case the constraint (27d) is not saturated. Therefore two constraints (27b) and (27c) must be saturated.

$$\begin{cases} q_1 = d_1; \\ q_2 = d_2; \end{cases} \Leftrightarrow \begin{cases} q_3 = \frac{d_1}{\beta_1}; \\ q_3 = \frac{d_2}{1-\beta_1}; \end{cases}$$

An example of optimal solution of the problem (27) in the (β_1, q_3) plane is shown in Figure 3.

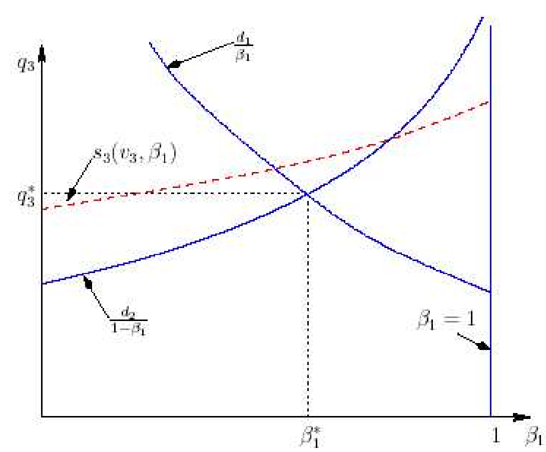


Figure 3: Optimal solution (β_1^*, q_3^*) in the (β_1, q_3) plane (Subcase 1.1 - Case $w_1 > w_2$).

4.1.2 Case 2: $q_3 = s_3(v_3, \beta_1)$.

Subcase 2.1

$$\begin{cases} q_1 < d_1; \\ q_2 < d_2; \end{cases} \Leftrightarrow \begin{cases} q_3 < \frac{d_1}{\beta_1}; \\ q_3 < \frac{d_2}{1-\beta_1}; \end{cases}$$

We draw in Figure 4 an example of optimal solution of the problem (27) in the (β_1, q_3) plane.

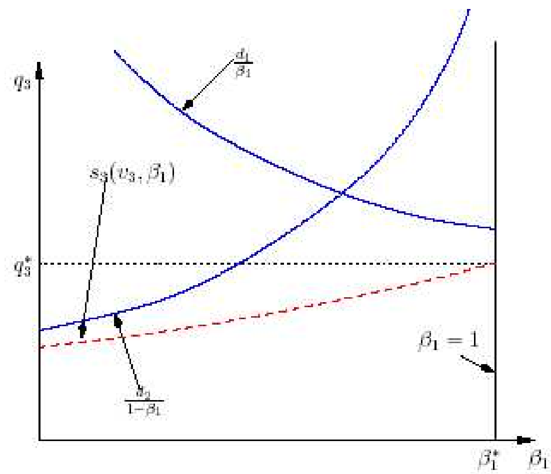


Figure 4: Optimal solution (β_1^*, q_3^*) in the (β_1, q_3) plane (Subcase 2.1 - Case $w_1 > w_2$).

Subcase 2.2

$$\begin{cases} q_1 = d_1; \\ q_2 < d_2; \end{cases} \Leftrightarrow \begin{cases} q_3 = \frac{d_1}{\beta_1}; \\ q_3 < \frac{d_2}{1-\beta_1}; \end{cases}$$

An example of optimal solution of the problem (27) is drawn is Figure 5.

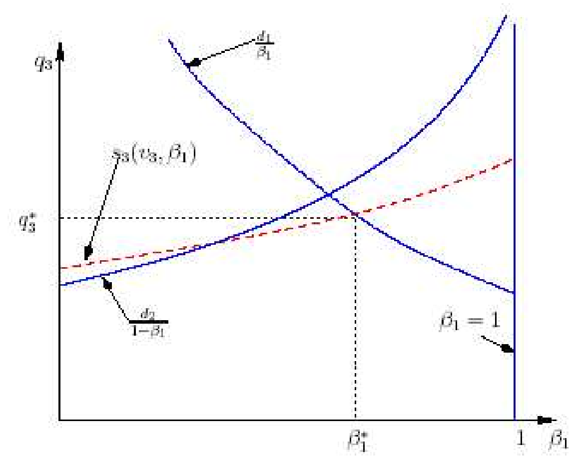


Figure 5: Optimal solution (β_1^*, q_3^*) in the (β_1, q_3) plane (Subcase 2.2 - Case $w_1 > w_2$).

Subcase 2.3

$$\begin{cases} q_1 < d_1; \\ q_2 = d_2; \end{cases}$$

In this subcase $q_1 < d_1$. Then, since $w_1 > w_2$, the drivers on road 1 are "more agressive" than those on road 2, so the flux on road 1 would strictly increase while the road 3 is not saturated (i. e. $q_3 = s_3(v_3, \beta_1)$). So, in this case we have necessarily $\beta_1 = 1$, i.e. $q_2 = (1 - \beta_1)q_3 = 0 =$. Since $q_2 = d_2$ then $d_2 = 0$. Therefore the road 2 is empty. This situation is in fact the case of one road with two different traffic conditions [2].

Subcase 2.4

$$\begin{cases} q_1 = d_1; \\ q_2 = d_2; \end{cases} \Leftrightarrow \begin{cases} q_3 = \frac{d_1}{\beta_1}; \\ q_3 = \frac{d_2}{1-\beta_1}; \end{cases}$$

In this subcase, the problem (27) has a unique solution. We give in Figure 6 an example of optimal solution of the problem (27).

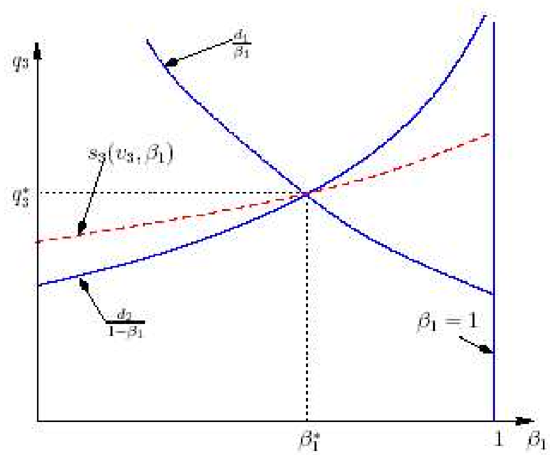


Figure 6: Optimal solution (β_1^*, q_3^*) in the (β_1, q_3) plane (Subcase 2.4 - Case $w_1 > w_2$)).

4.2 The case $w_1 = w_2$

The optimal solution of the problem (27) is reached only in one of the following cases:

4.2.1 Case 1: $q_3 < s_3(v_3, \beta_1)$

Subcase 1.1

$$\begin{cases} q_1 = d_1; \\ q_2 = d_2; \end{cases} \Leftrightarrow \begin{cases} q_3 = \frac{d_1}{\beta_1}; \\ q_3 = \frac{d_2}{1-\beta_1}; \end{cases}$$

In this subcase the problem (27) has a unique solution.
An example of optimal solution of the problem (27) is drawn is Figure 7.

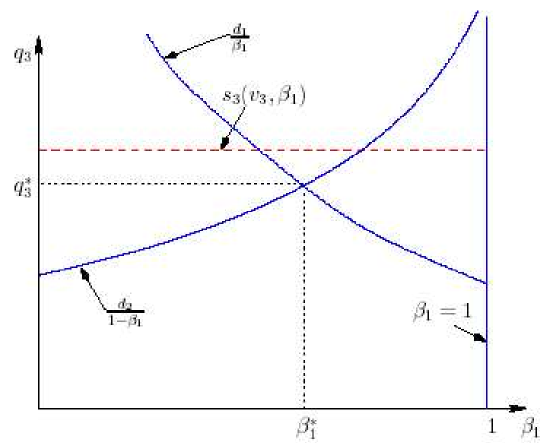


Figure 7: Optimal solution (β_1^*, q_3^*) in the (β_1, q_3) plane (Subcase 1.1 - Case $w_1 = w_2$).

4.2.2 Case 2: $q_3 = s_3(v_3, \beta_1)$

Subcase 2.1

$$\begin{cases} q_1 = d_1; \\ q_2 = d_2; \end{cases} \Leftrightarrow \begin{cases} q_3 = \frac{d_1}{\beta_1}; \\ q_3 = \frac{d_2}{1-\beta_1}; \end{cases}$$

As in the subcase 1.1 above, the problem (27) has a unique solution.
An example of optimal solution of the problem (27) is drawn is Figure 8.

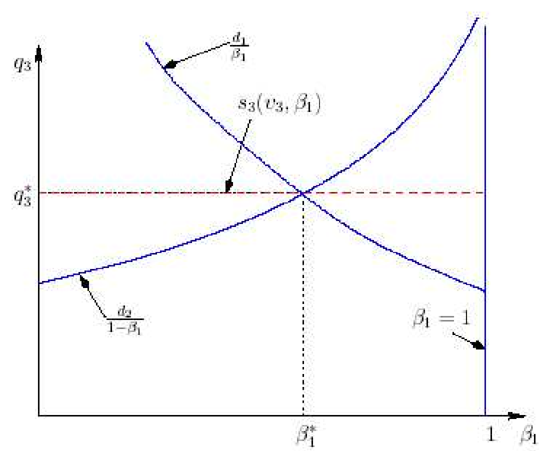


Figure 8: Optimal solution (β_1^*, q_3^*) in the (β_1, q_3) plane (Subcase 2.1 - Case $w_1 = w_2$).

Subcase 2.2

$$\begin{cases} q_1 \le d_1; \\ q_2 \le d_2; \\ \text{with}(q_1, q_2) \ne (d_1, d_2) \end{cases}$$

In this subcase the problem (27) has **an infinity** of solutions.
An example of optimal solution of the problem (27) is drawn is Figure 9.

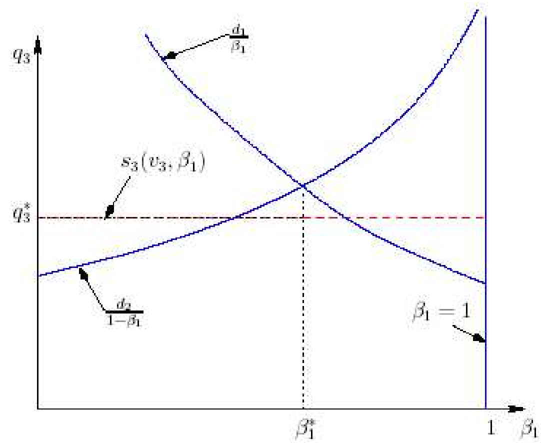


Figure 9: Optimal solution (β_1^*, q_3^*) in the (β_1, q_3) plane (Subcase 2.2 - Case $w_1 = w_2$).

In the Appendix we give a numerical algorithm which solves (27) when the solution is unique. There, for simplicity, when there is **no** uniqueness, we fix $\beta_1 = \beta_1^{**} := \frac{d_1}{d_1+d_2}$ as **additional** assumption.

Finally, we summarize the discussion above in the following proposition.

Proposition 4.1. *Consider two incoming roads* $1, 2$ *and one outgoing road* 3 *with* $a_1 = a_2 = -\infty, b_1 = b_2 = a_3$ *and* $b_3 = \infty$ *and constant initial data* $U_{i,0} = (\rho_{i,0}\rho_{i,0}v_{i,0}), i = 1, 2, 3$. *Assume* $\mathbf{w}_1(U_{1,0}) \ne \mathbf{w}_2(U_{2,0})$.

Then there exists a unique solution $\{U_i(x,t)\}_{i=1,2,3}$ *of the Riemann problem at the junction (11) and (12) with the following properties:*

1. $\{U_i(x,t)\}_{i=1,2,3}$ *is a weak solution of the network problem (7-8), where* $p_i^\dagger \equiv p_i$ *for the incoming roads* $i = 1, 2$.

   For the outgoing road $i = 3$, *we obtain two different expressions for* $p_3^\dagger$, *depending on the position* (x,t):

   In the $x - t$ *plane, in a triangle near the junction, we consider the* homogenized solution *described above. Therefore,* $p_i^\dagger(\cdot) := p_i^*(\cdot)$ *is given by the general relations (16), (18), (19) and (20)and more precisely by formulas (23)–(26). The triangle is bounded at any fixed time* $t > 0$ *by* $x = a_3$ *and* $x = a_3 + tv_{3,0}$.

   In the remaining part of the outgoing road we have $p_3^\dagger \equiv p_3$.

2. *The Rankine-Hugoniot conditions (9a-9b) are satisfied, with* $\rho_i(x,t)v_i(x,t) \geq 0\,,\ 1 \leq i \leq 3$. *In particular* $U_3(a_3^+,t)$ *satisfies*

$$\mathbf{w}_3^\dagger(U_3(a_3^+,t)) :=\ \mathbf{w}_3^*(U_3(a_3^+,t)) := v_3(a_3^+,t) + p_3^*(\rho_3(a_3^+,t)) = \bar{w},$$

where $\bar{w}$ *is the* homogenized value *given by (16).*

3. *The incoming fluxes are maximal subject to the other conditions.*

We now consider a second type of junction (a diverge).

5 One incoming road and two outgoing roads

Here, we follow the presentation of [12]. The results are also recovered by the presentation in [9]: they are just recalled for sake of completeness.

In this case, $i = 1$ for the incoming road and $j = 2,3$ for the outgoing roads. For notational convenience we set $\alpha_{21} = \alpha$ and $\alpha_{31} = (1-\alpha)$. Furthermore, we set $w_1 := \mathbf{w}_1(U_{1,0})$.

Again, we simplify the general maximization problem (21). From equation (22d), we obtain

$$\beta_{j1} = \frac{\alpha_{j1}q_1}{q_j}, \qquad j = 2,3.$$

Since there is only one incoming road, $q_j = \alpha_{j1}q_1$, $j = 2,3$ and therefore $\beta_{21} = \beta_{31} = 1$.

Obviously, here, **no** homogenization is needed, since there is a single incoming road: all the cars have kept the same value w_1 (”color”) when passing the junction. Hence, $p_j^\dagger \equiv p_j$ for $j = 2,3$ and

$$\mathbf{w}_j(U) = v + p_j(\rho_j) = v + P_j(\tau_j) = w_1 \quad j = 2,3. \tag{30}$$

As above, we assume that for each $k = 1,2,3$, $p_k(\rho) = \rho^\gamma$ (or equivalently $P_k(\tau) = 1/\tau^\gamma$) and $\gamma = 1$.

Now, we discuss the possible supplies $s_j(U_{j,0})$ and then finally solve (21). By equation (30) we have

$$\rho_j v_j = v_j(w_1 - v_j), j = 2,3.$$

Let $v_{j,c}$ the velocity corresponding to the maximal flux on the outgoing road j, i.e.,

$$v_{jc} = \{v_j; \quad \frac{d(\rho_j v_j)}{dv_j} = 0\} = \frac{w_1}{2} \qquad j = 2,3$$

Therefore, the supplies $s_j(U_{j,0}; \mathbf{w}_j, w_1)$ are

$$s_j(U_{j,0}; \mathbf{w}_j, w_1) = \begin{cases} v_{j,0}(w_1 - v_{j,0}) & if \ \ v_{j,0} \leq \frac{w_1}{2}; \\ \\ \frac{w_1}{2}(w_1 - \frac{w_1}{2}) & if \ \ v_{j,0} > \frac{w_1}{2}. \end{cases} \tag{31}$$

Finally, problem (22) reduces to

$$\max q_1 \text{ subject to} \tag{32a}$$
$$0 \le q_1 \le d_1, \tag{32b}$$
$$0 \le \alpha q_1 \le s_2(U_{2,0}; \mathbf{w}_2, w_1), \tag{32c}$$
$$0 \le (1-\alpha) q_1 \le s_3(U_{3,0}; \mathbf{w}_3, w_1), \tag{32d}$$

and its optimal solution is $q_1^* = \min\{d_1, \frac{s_2(U_{2,0};\mathbf{w}_2,w_1)}{\alpha}, \frac{s_3(U_{3,0};\mathbf{w}_3,w_1))}{1-\alpha}\}$.

As before, the above can be summarized in the following proposition, c.f. Proposition 4.1 [12].

Proposition 5.1. *Consider three roads $i = 1,2,3$ with $a_1 = -\infty, b_1 = a_2 = a_3$ and $b_2 = b_3 = \infty$ and constant initial data $U_{i,0} = (\rho_{i,0}, \rho_{i,0} v_{i,0}), i = 1,2,3$. Let $0 \le \alpha \le 1$ be given.*

Then there exists a unique solution $\{U_i(x,t)\}_{i=1,2,3}$ of the Riemann problem at the junction (11) and (12) with the following properties:

1. *$\{U_i(x,t)\}_{i=1,2,3}$ is a weak solution of the network problem (7-8) with $p_i^\dagger \equiv p_i$.*
   Therefore, the Rankine-Hugoniot conditions (9a-9b) are satisfied, and $\rho_i(x,t) v_i(x,t) \ge 0, \ , \ i = 1,2,3$.

2. *For all $t > 0$ the flux is distributed in proportions α and $1-\alpha$ between roads 2 and 3, c.f. equations (21e):*

$$\alpha(\rho_1 v_1)(b_1^-, t) = (\rho_2 v_2)(a_2^+, t), \tag{33a}$$
$$(1-\alpha)(\rho_1 v_1)(b_1^-, t) = (\rho_3 v_3)(a_3^+, t), \tag{33b}$$

3. *The flux $(\rho_1 v_1)(b_1^-, t)$ is maximal at the interface, subject to the above conditions, c.f. equation (21a).*

6 Extensions and Numerical Results

We present a modelling to deal with a general junction with n incoming and m outgoing roads. Again, we have to solve the general maximization problem (21). But, due to the strong non–linearities arising in particular in equation (21c), this is more complex. However, most of the traffic intersections with n incoming and m outgoing roads can be seen and (in fact are designed) as roundabouts. Due to the geometry of roundabouts (see Figure 10), all the *conflict points* (i.e. points of intersections of roads) are either $2 \mapsto 1$ or $1 \mapsto 2$ junctions, which have been discussed above. Therefore, we model a general junction by a combination of the two types of junctions discussed above. An example is given in Figure 10. Finally, we present numerical results on the $2 \mapsto 1$ junction. We use the algorithm presented in Section (A) to solve the maximization problem (27) for constant initial data $U_{i,0}$ to obtain the optimal β. The numerical simulation is obtained by applying a standard first–order relaxation scheme [14] with fixed discretization $\Delta x = 1/800$. We set the following initial data $U_{1,0} = [2,3]$, $U_{2,0} = [3,5]$ and $U_{3,0} = [3,7]$ and use $p_i(\rho) = p(\rho)$ on all roads $j = 1,2,3$.

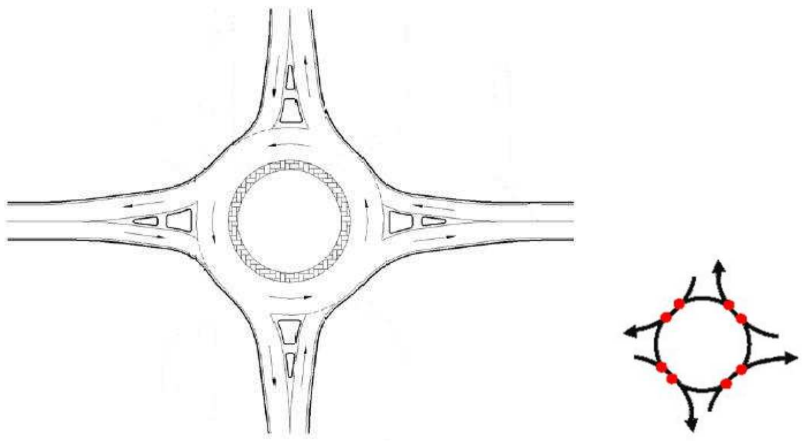


Figure 10: Roundabouts for a 4–4 junction

Then, the optimal β is $\beta^{opt} = 0\%$, the maximal flux at the interface is $q^{opt} = 49/9$ and the states $\bar{U}_i$ are $\bar{U}_1 = [7/2, 0.], \bar{U}_2 = [7/3, 5.44]$ and $\bar{U}_3 = [7/3, 5.44]$. Initially the drivers on road $j = 2$ are assumed to be more "aggressive", in the sense that $w_1(U_{1,0}) < w_2(U_{2,0})$. Furthermore, the supply on road $j = 3$ is not sufficient to collect *both* inflows. Hence, the maximization of the flux (27) yields a decrease in the flux on road $j = 1$ and let all the flux on road $j = 2$ pass the junction. Therefore, we finally observe $\bar{q}_2 = \bar{q}_3$. On the incoming road $j = 1$ we connect $U_{1,0}$ and $\bar{U}_1$ by a 1–shock wave of negative speed and this is also seen in Figure 11. The 1–shock wave corresponds to a braking of the cars on road $j = 1$ while they enter the junction. Furthermore, one the incoming road $j = 2$ we have $U_{2,0}$ connected to $\bar{U}_2$ by a 1–rarefaction, see also Figure 12. Here, the more aggressive drivers (compared with road one) accelerate in order to enter the intersection and pass through to road $j = 3$. On the outgoing road $\bar{U}_3$ and $U_{3,0}$ by a 2–contact discontinuity. The flux has increased up to the maximal possible flux on this road due to the maximization (27). Flux and snapshots of the solution on the outgoing road are depicted Figure 13.

7 Summary

We extended the work of [12] and obtain a general formulation for suitable coupling conditions at the intersection *without* imposing a fixed mixture principle. The final solution conserve mass and (pseudo-)momentum and additionally maximizes the total flux at an intersection. This has been fully solved in the case of $1 \mapsto 2$ and $2 \mapsto 1$ junctions and numerical results for (the more interesting) later case have been given. Finally, we present a simple modelling of an $n \mapsto m$ junction.

Acknowledgements

The presented work has been partially supported by the Kaiserslautern Excellence Cluster "Dependable Adaptive Systems and Mathematical Modelling", by the former HYKE N^o HPRN-CT-2002-00282 and by the French ACI-NIM (Nouvelles Interactions des Mathématiques) N^o 193 (2004).

References

[1] A. Aw, A. Klar, M. Materne and M. Rascle, *Derivation of continuum flow traffic models from microscopic follow the leader models*, SIAM J. Appl. Math., **63** (2002), p. 259.

[2] A. Aw and M. Rascle, *Resurection of second order models of traffic flow?*, SIAM J. Appl. Math., **60** (2000), p. 916.

[3] P. Bagnerini and M. Rascle, *A multi-class homogenized hyperbolic model of traffic flow*, SIAM J. Math. Anal., **35** 2003, **N 4**, 949-973 .

[4] C. Bardos, A. LeRoux, and J. Nedelec, *First order quasilinear equations with boundary conditions*, Comm. in partial differential equations, 4 (1979), p. 1017.

[5] G. Coclite, M. Garavello, and B. Piccoli, *Traffic flow on road networks*, SIAM J. Math. Anal., 36 (2005), p. 1862

[6] C. M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics*, Springer Verlag, Berlin, Heidelberg, New York, 2000.

[7] C. F. Daganzo, *Requiem for second order fluid approximations of traffic flow*, Trans. Res. B, 29 (1995), p. 277.

[8] ——, *A behavioral theory of multi-lane traffic flow part I: Long homogeneous freeway sections*, Trans. Res. B, 36 (2002), p. 131.

[9] M. Garavello and B. Piccoli, *Traffic flow on a road network using the "Aw-Rascle" model*, to appear, (2005).

[10] J. Greenberg, *Extension and amplification of the Aw-Rascle model*, SIAM J. Appl. Math., 63 (2001), p. 729.

[11] M. Herty and A. Klar, *Modelling and optimization of traffic networks*, SIAM J. Sci. Comp., 25 (2004), p. 1066.

[12] M. Herty and M. Rascle *Coupling conditions for a class of "second-order" models for traffic flow*, Preprint, (2005).

[13] H. Holden and N. H. Risebro, *A mathematical model of traffic flow on a network of unidirectional roads*, SIAM J. Math. Anal., 26 (1995), p. 999.

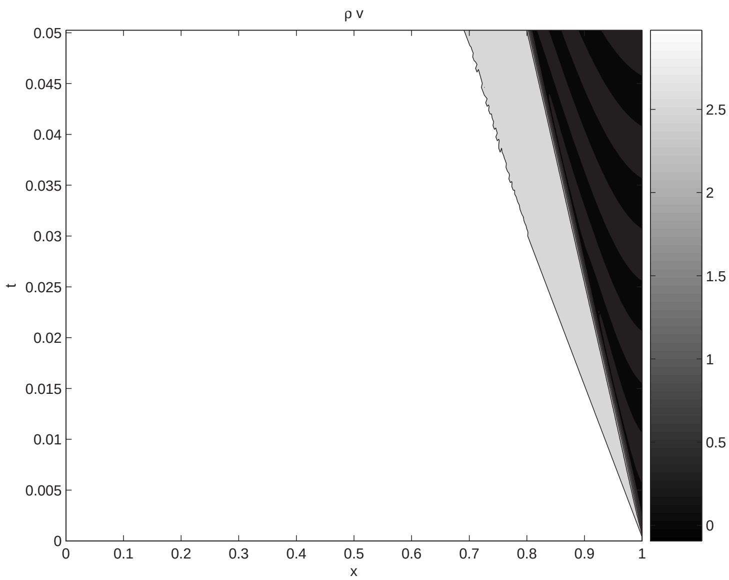
[14] S. Jin and Z. Xin, *The Relaxation Schemes for Systems of Conservation Laws in Arbitrary Space Dimensions*, Comm. Pure Appl. Math., 48 (1995), p. 235,

[15] J. P. Lebacque, *Les modeles macroscopiques du traffic*, Annales des Ponts., 67 (1993), pp. 24–45.

[16] M. Lighthill and J. Whitham, *On kinematic waves*, Proc. Royal Soc. Edinburgh, A229 (1955), p. 281.

[17] H. Payne, *FREFLO: A macroscopic simulation model for freeway traffic*, Transportation Research Record, 722 (1979), p. 68.

[18] H. M. Zhang, *A non-equilibrium traffic model devoid of gas-like behaviour*, Trans. Res. B, 36 (2002), p. 275.



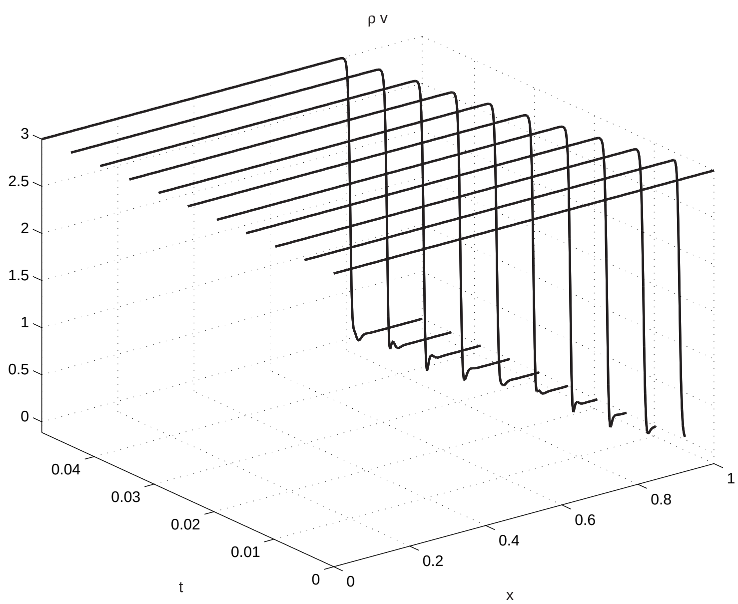


Figure 11: Plot of the contour line of the flux $\rho_1 v_1$ (top) and snapshots of the solution for some times t (bottom).

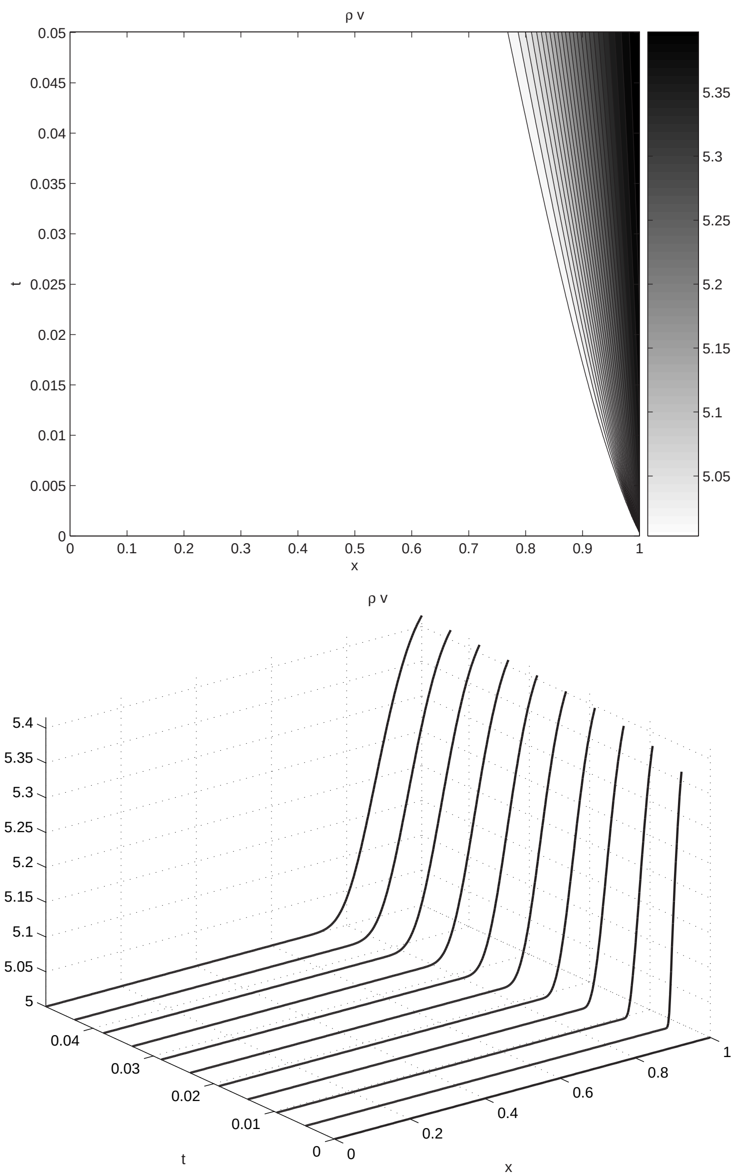


Figure 12: Plot of the contour line of the flux $\rho_2 v_2$ (top) and snapshots of the solution for some times t (bottom).

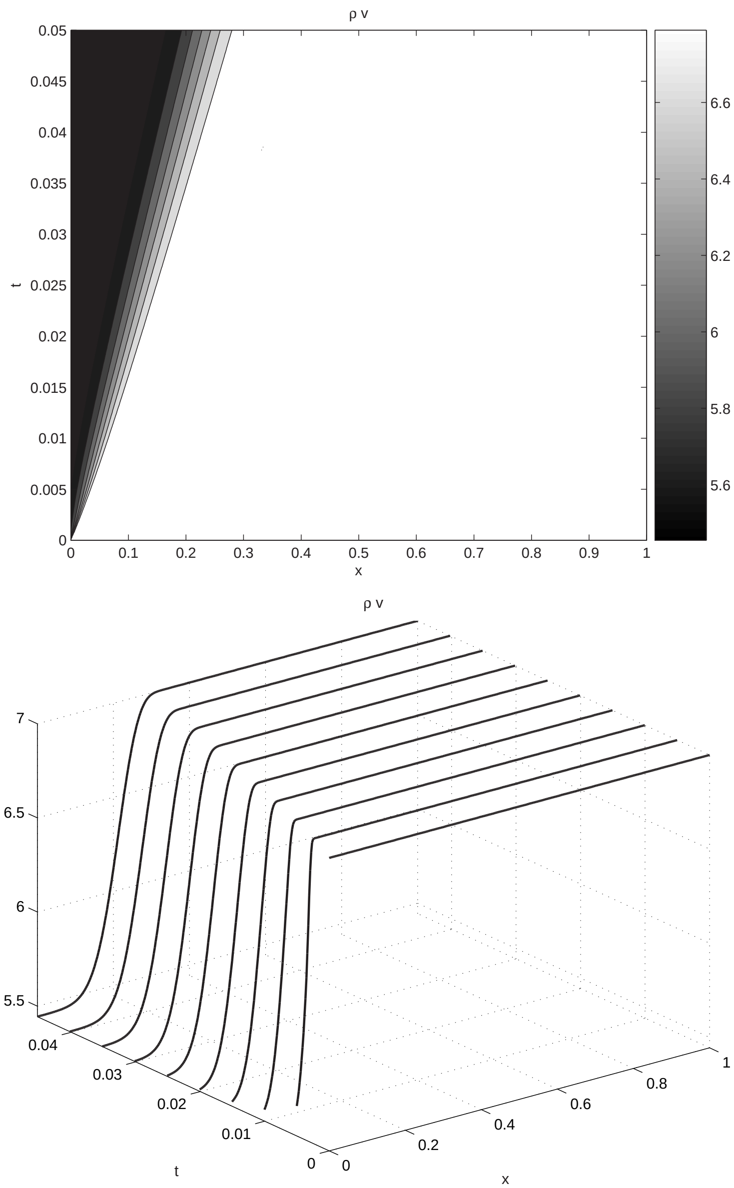


Figure 13: Plot of the contour line of the flux $\rho_3 v_3$ (top) and snapshots of the solution $\rho_3 v_3$ for some times t (bottom).

A Algorithm for solving (27)

Algorithm
Begin
If $w_1 > w_2$ **then**
If $s_3(v_3, 1) \leq d_1$ **then**
$q_3^* := s_3(v_3, 1);$
$\beta_1^* := 1;$
Else
$\beta_1^{**} := \{\beta_1 / \ \frac{d_1}{\beta_1} = \frac{d_2}{1-\beta_1}\}; \ (\beta_1^{**} := \frac{d_1}{d_1+d_2})$
$\beta_1^* := \{\beta_1 / \ \ s_3(v_3, \beta_1) = \frac{d_1}{\beta_1}\};$
If $\beta_1^* \geq \beta_1^{**}$ **then**
$q_3^* := s_3(v_3, \beta_1^*);$
Else
$q_3^* := s_3(v_3, \beta_1^{**});$
$\beta_1^* := \beta_1^{**};$
EndIf
EndIf
Else $(w_1 \leq w_2)$
If $w_1 < w_2$ **then**
If $s_3(v_3, 0) \leq d_2$ **then**
$q_3^* := s_3(v_3, 0);$
$\beta_1^* := 0;$
Else
$\beta_1^{**} := \{\beta_1 / \ \frac{d_1}{\beta_1} = \frac{d_2}{1-\beta_1}\}; \ (\beta_1^{**} := \frac{d_1}{d_1+d_2})$
$\beta_1^* := \{\beta_1 / \ \ s_3(v_3, \beta_1) = \frac{d_2}{1-\beta_1}\};$
If $\beta_1^* \leq \beta_1^{**}$ **then**
$q_3^* := s_3(v_3, \beta_1^*)$
Else
$q_3^* := s_3(v_3, \beta_1^{**});$
$\beta_1^* := \beta_1^{**};$
EndIf
EndIf
Else $(w_1 = w_2)$
$\beta_1^{**} := \{\beta_1 / \ \frac{d_1}{\beta_1} = \frac{d_2}{1-\beta_1}\};(\beta_1^{**} := \frac{d_1}{d_1+d_2})$
If $s_3(v_3, \beta_1^{**}) > \frac{d_1}{\beta_1^{**}}$ **then** (or $s_3(v_3, \beta_1^{**}) > \frac{d_2}{1-\beta_1^{**}}$)
$\beta_1^* := \beta_1^{**};$
$q_3^* := \frac{d_1}{\beta_1^{**}};$ (or $q_3^* := \frac{d_2}{1-\beta_1^{**}}$)
Else
$\beta_1^* := \beta_1^{**};$
$q_3^* := s_3(v_3, \beta_1^{**})$
EndIf
EndIf
EndIf
RETURN(q_3^*, β_1^*)
End