Asymptotic and Discrete Concepts for Optimal Control in Radiative Transfer

Michael Herty* **Rene Pinnau†** **Guido Thömmes‡**

February 28, 2006

Abstract

Optimal control problems for radiative transfer and for approximate models are considered. Following the approach *first discretize, then optimize*, the discrete SP_N approximations are for the first time derived exactly, which can be used to study optimal control on reduced order models. Combining asymptotic analysis and the adjoint calculus yields diffusive-type approximations for the adjoint radiative transport equation in the spirit of the approach *first optimize, then discretize.*

Keywords. Radiative transfer, Optimal control, First-Order Optimality, Discrete ordinates, SP_N-equations, Iterative methods, Adjoint calculus.
AMS(MOS) subject classification. 35K55, 49K20, 80A20.

1 Introduction

The simulation and optimization of radiation dominated heat transfer processes, e.g. in the field of glass cooling processes or the design of combustion chambers, pose several challenging problems for applied mathematicians and engineers and have been extensively studied during the last years [3, 21, 16, 2, 1, 6, 13]. Due to the high numerical complexity of the model equations which are given by the radiative heat transfer system for the radiative intensity, there is a strong need for sophisticated reduced order models. This led to the development of a whole hierarchy of approximate equations, ranging from moment expansions to diffusive type models, like the SP_N-systems [9, 8, 4]. Other recent developments discuss the coupling of radiative transfer and heat equations, see for example [7, 10], and efficient numerical schemes for resolving the arising systems [17, 19, 18]. In this work, we are interested in the study of *optimal control problems* of tracking-type in radiative transfer. The general problem we are considering is formulated as follows:

$$\min_{\varphi,Q} \mathcal{F}(\varphi, Q) \text{ subject to (1.3)}, \tag{1.1}$$

*Fachbereich Mathematik, Technische Universität Kaiserslautern, D–67663 Kaiserslautern, Germany (herty@rhrk.uni-kl.de)
†Fachbereich Mathematik, Technische Universität Kaiserslautern, D-67663 Kaiserslautern, Germany (pinnau@mathematik.uni-kl.de)
‡Fachbereich Mathematik, Technische Universität Kaiserslautern, D-67663 Kaiserslautern, Germany (thoemmes@mathematik.uni-kl.de)

where φ is the mean intensity,

$$\varphi(\mathbf{x}) := \int_{S^{d-1}} \mathrm{I}(\mathbf{x}, \omega) d\omega, \tag{1.2}$$

S^{d-1} the unit sphere in $\mathbb{R}^d$, and $\mathrm{I} = I(\mathbf{x}, \omega)$, the intensity at position $\mathbf{x}$ propagating along direction ω. We assume, I satisfies the scaled radiative transfer equation

$$\begin{aligned} \varepsilon\omega \cdot \nabla \mathrm{I} + (\sigma + \kappa)\mathrm{I} &= \frac{\sigma}{4\pi} \int_{S^{d-1}} \mathrm{I} d\omega + Q, \\ \mathrm{I}(\omega, \mathbf{x}) &= A, \qquad \mathbf{n} \cdot \omega < 0, \end{aligned} \tag{1.3}$$

for given boundary data $A = A(\omega)$ and source term $Q = Q(\mathbf{x})$. Further, $\kappa = \kappa(\mathbf{x})$ is the absorption coefficient and $\sigma = \sigma(\mathbf{x})$ is the scattering coefficient. By $n = n(\mathbf{x})$ we denote the outer normal at a boundary point $\mathbf{x}$. We consider an optically thick medium, where the opacity or the scattering cross section are large and therefore the radiation is conveyed in a diffusive way. Hence, the scaled equation contain the asymptotic parameter $\varepsilon = 1/((\kappa_{ref} + \sigma_{ref})\mathbf{x}_{ref})$ which tends to 0.

Finally, the objective functional is of tracking-type,

$$\mathcal{F}(\varphi, Q) := \frac{\alpha_1}{2} \int_D (\varphi - \bar{\varphi})^2 \, d\mathbf{x} + \frac{\alpha_2}{2} \int_D \left(Q - \bar{Q}\right)^2 d\mathbf{x}, \tag{1.4}$$

for a domain $D \in \mathbb{R}^d$ and positive constants α_1, α_2 and given (desired) states and controls $\bar{\varphi}(\mathbf{x})$ and $\bar{Q}(\mathbf{x})$, respectively.

This work extends recently published results [5, 14] on such optimal control problems (1.1). Here, we essentially focus on the well-known SP_N approximations for (1.3) (see e.g. [9, 8] and the references therein). We employ techniques from asymptotic analysis to derive approximate models and adjoints which are more suitable for optimization purposes than the full transport equation. Essentially, we follow the two common approaches *first discretize, then optimize*, or, vice versa, *first optimize, then discretize*.

Our motivation is two-fold: First, we are interested in a thorough derivation of the SP_N approximations by starting from a discretization of the radiative transfer equations and using von Neumann's series to invert the arising matrices of the discrete system. This procedure parallels the derivation of the SP_N approximations in the continuous case and leads to discretizations of the corresponding diffusion equations, c.f. Section 2.

Second, we apply the adjoint calculus to the RHT equation and use the continuous SP_N approximation procedure to derive the SP_N system for the adjoint problem, see Section 3. In combination with the results of [5, 14] we finally establish the following diagram (Figure 1) for general tracking-type optimal control problems in radiative transfer.

We collect some references concerning already discussed connections in Figure 1:

- 'Continuous RTE' $\rightarrow$ 'SP_N approximations': The continuous SP_N approximations for the continuous RT equation are well-known, see for example [20, 9]. But they can be only derived in a *formal* manner due the usage of the von Neumann series for unbounded operators.

- 'Continuous RTE' $\rightarrow$ 'Continuous Adjoints of the RTE': This connection has been established in [5]. The result, which will be used later, is as follows [5, Theorem 1]: Given

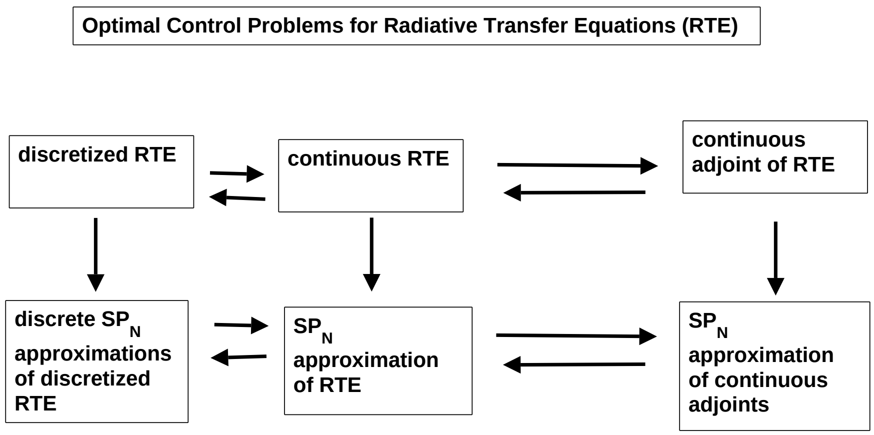

Figure 1: Diagram of the interplay of discrete, adjoint and SP_N approximations in radiative transfer.

a sufficiently regular, bounded domain $D \in \mathbb{R}^{2,3}$ and $\bar{\varphi}, \bar{Q} \in L^2(D)$. Then, there exists a function $\lambda \in L^2(D)$, such that the necessary and sufficient first-order optimality conditions for (1.1). Moreover, λ could be obtained from the formal adjoints to (1.3), i.e., λ dependents on the solution J to

$$\begin{aligned} -\varepsilon\omega \cdot \nabla \mathrm{J} + (\sigma+\kappa)\mathrm{J} &= \frac{\sigma}{4\pi}\int_{S^{d-1}} \mathrm{J} d\omega + \alpha_1 \left(\varphi(\mathbf{x}) - \bar{\varphi}(\mathbf{x})\right), \\ \mathrm{J}(\omega, \mathbf{x}) &= 0, \qquad \mathbf{n}\cdot\omega > 0, \end{aligned} \tag{1.5}$$

and the following (gradient) equation holds:

$$\alpha_2 \left(Q(\mathbf{x}) - \bar{Q}(\mathbf{x})\right) + \int_{S^{d-1}} \mathrm{J} d\mathbf{x} = 0. \tag{1.6}$$

- 'SP_N approximations of RTE' $\rightarrow$ 'SP_N of adjoints of the RTE' as been discussed in [14, 15]. Here, the adjoint calculus was used for the derivation of first- and second-order optimality conditions for minimization problems constrained by the continuous SP_N system coupled with a nonlinear heat equation.

In this work, we discuss the remaining links, being 'discretized RTE' $\rightarrow$ 'discrete SP_N, 'discrete SP_N' $\rightarrow$ 'SP_N of continuous RTE' (both in Section 2) and finally, 'adjoints of RTE' $\rightarrow$ 'SP_N of adjoints' (see Section 3).

Remark 1 *In general, radiative transfer equations have the form [12, 11]*

$$\varepsilon\omega \cdot \nabla \mathrm{I} + (\sigma + \kappa)\mathrm{I} = \frac{\sigma}{4\pi} \int_{S^{d-1}} \mathrm{I} d\omega + \kappa B(\theta). \tag{1.7}$$

In (1.7), θ is the temperature, and $B(\theta)$ is the intensity of the black-body radiation given by the Stefan-Boltzmann law, compare [11] for further physical details. The equations (1.7) are solved subject to the boundary condition

$$\mathrm{I}(t\omega, \mathbf{x}) = \varrho(\mathbf{n} \cdot \omega)\mathrm{I}(t, \omega', \mathbf{x}) + (1 - \varrho(\mathbf{n} \cdot \omega))\, B(\theta_b), \quad \mathbf{n} \cdot \omega < 0, \tag{1.8}$$

where $\varrho \in [0, 1]$ is the reflectivity: $\varrho = 1$ for total reflection and $\varrho = 0$ for Dirichlet type boundary conditions. The temperature θ_b is a fixed boundary function, $\mathbf{n}$ is the outward normal on the boundary and $\omega' = \omega - 2(\mathbf{n} \cdot \omega)\mathbf{n}$ is the specular reflection of ω on the boundary.

For the sake of simplicity, we shall consider the radiative transfer model (1.3). Although we assume several simplifications, these equations are still reasonable in many applications, e.g. when dealing with a grey medium and if the mean free path of the radiation is small compared to a characteristic length.

2 Discrete derivation of the SP_N approximations

In this section we follow the approach *first discretize, then optimize* to derive a reduced order model valid in the diffusive limit on the discrete level. Due to the special, symmetric structure of the discretizations these can then be directly used for the implementation of the corresponding discrete adjoints.

Starting from a discretization of the radiative transfer equations, we use Neumann's series to invert the matrices of the discrete system and derive a discrete version of the SP_N approximations. The procedure closely resembles the derivation of the SP_N approximations in the continuous case and results in valid discretizations of the corresponding diffusion equations. This substantiates the construction of the continuous SP_N equations in the framework of an asymptotic analysis where Neumann's series has been formally applied to invert the transport operator, which contains the unbounded space derivative operator [8] .

The medium is assumed to be optically thick, i.e. κ or σ are large such that the mean free path is small. For notational convenience we define the source term

$$\kappa B := Q. \tag{2.1}$$

The continuous SP_1, SP_2 and SP_3 approximations of (1.3) are given, respectively, by

$$-\frac{\varepsilon^2}{3(\sigma + \kappa)} \nabla^2 \varphi + \kappa\varphi = 4\pi\kappa B, \tag{2.2}$$

and

$$-\varepsilon^2 \frac{5(\sigma + \kappa) + 4\kappa}{15(\sigma + \kappa)} \nabla^2 \xi + \kappa\xi = 4\pi\kappa B, \tag{2.3}$$

where $\xi = \varphi + \frac{4\kappa}{5(\sigma+\kappa)}(\varphi - 4\pi B)$, and

$$-\frac{\varepsilon^2}{3(\sigma + \kappa)} \nabla^2 (\varphi + 2\varphi_2) + \kappa\varphi = 4\pi\kappa B, \tag{2.4a}$$

$$-\frac{9\varepsilon^2}{35(\sigma + \kappa)} \nabla^2 \varphi_2 + (\sigma + \kappa)\varphi_2 - \frac{2}{5}\kappa\varphi = -\frac{2}{5}\, 4\pi\kappa B. \tag{2.4b}$$

In all cases φ is the mean intensity given by (1.2). For the validity of the SP_N approximations we refer to [9],[8] and [20].

For our analysis we consider the radiative transfer equations in one-dimensional slab geometry

$$\varepsilon\mu\partial_x I + (\sigma+\kappa)I = \frac{\sigma}{2}\int_{-1}^{+1} I(\mu')d\mu' + \kappa B,$$

with angular variable $\mu \in [-1,+1]$. An upwind discretization for the unknown intensities I_{km} on the equidistant grid $x_k = kh$, $k = 0,...,K$ with step size h and an angular discretization based on a even order quadrature formula with nodes $\mu_1 < ... < \mu_{\frac{M}{2}} < 0 < \mu_{\frac{M}{2}} < ... < \mu_M$ on the interval $[-1,1]$ with weights w_m is given by

$$\mu_m > 0: \quad I_{0m} = \hat{I}^L_m, \tag{2.5a}$$

$$\varepsilon\mu_m\frac{I_{km} - I_{k-1,m}}{h} + (\sigma+\kappa)I_{km} = \frac{\sigma}{2}\sum_{m'=1}^{M} w_{m'}I_{km'} + \kappa B_k, \tag{2.5b}$$

$$\mu_m < 0: \quad I_{Km} = \hat{I}^R_m, \tag{2.5c}$$

$$\varepsilon\mu_m\frac{I_{k+1,m} - I_{km}}{h} + (\sigma+\kappa)I_{km} = \frac{\sigma}{2}\sum_{m'=1}^{M} w_{m'}I_{km'} + \kappa B_k. \tag{2.5d}$$

We investigate asymptotic approximations to these equations in the diffusive limit where the asymptotic parameter ε tends to 0.

In matrix-vector form the linear systems for vectors $I_m = (I_{km})_{k=0,...,K}$ read

$$\mu_m > 0: (Id + \frac{\varepsilon\mu_m}{\bar{\kappa}h}D_m)I_m = \frac{\kappa}{\bar{\kappa}}(\frac{\sigma}{2\kappa}\sum_{m'=1}^{M} w_{m'}I_{m'} + b_m) = c_m,$$

$$D_m = D_+ = \begin{bmatrix} 0 & & & & \\ -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{bmatrix}, \quad b_m = \begin{bmatrix} \hat{I}^L_m \\ B_1 \\ \vdots \\ B_{K-1} \\ B_K \end{bmatrix} = B + R^L_m$$

$$\mu_m < 0: (Id + \frac{\varepsilon|\mu_m|}{\bar{\kappa}h}D_m)I_m = \frac{\kappa}{\bar{\kappa}}(\frac{\sigma}{2\kappa}\sum_{m'=1}^{M} w_{m'}I_{m'} + b_m) = c_m$$

$$D_m = D_+ = \begin{bmatrix} 1 & -1 & & & \\ & \ddots & \ddots & & \\ & & 1 & -1 & \\ & & & 1 & -1 \\ & & & & 0 \end{bmatrix}, \quad b_m = \begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_{K-1} \\ \hat{I}^R_m \end{bmatrix} = B + R^R_m$$

where we set $\bar{\kappa} = \sigma + \kappa$ for notational convenience. The boundary conditions introduce perturbations $R^L_m = (\hat{I}^L_m - B_0, 0, ..., 0)^T$ and $R^R_m = (0, ..., 0, \hat{I}^R_m - B_K)^T$ of the right hand side B. When ε is sufficiently small, i.e. $\varepsilon < \bar{\kappa}h$, then the spectral radius of $\frac{\varepsilon|\mu_m|}{\bar{\kappa}h}D_m$ is less than 1 and Neumann's series can be applied to solve for I_m.

Remark 2 *Hence we have the condition $\varepsilon = O(h)$ for the applicability of the first step in the derivation of the SP_N equations. This reflects the fact that the onesided differences above discretize the unbounded space derivative operator, and the norms of the corresponding matrices grow with order $O(1/h)$ as h tends to zero.*

From the Neumann series representation of the intensity

$$\begin{aligned} I_m &= \big(Id + \frac{\varepsilon|\mu_m|}{\bar\kappa h} D_m\big)^{-1} c_m \\ &= \Big(Id - \frac{\varepsilon|\mu_m|}{\bar\kappa h} D_m + (\frac{\varepsilon|\mu_m|}{\bar\kappa h} D_m)^2 - (\frac{\varepsilon|\mu_m|}{\bar\kappa h} D_m)^3 \pm \dots\Big) c_m \end{aligned}$$

we compute the mean intensity $\varphi = (\varphi_k)_{k=0,\dots,K}$

$$\begin{aligned} \varphi &= 2\pi \sum_m w_m I_m \\ &= 2\pi \sum_m w_m \big(Id + \frac{\varepsilon|\mu_m|}{\bar\kappa h} D_m\big)^{-1} c_m \\ &= 2\pi \Big(\sum_m w_m c_m - \frac{\varepsilon h}{\bar\kappa} \sum_m w_m |\mu_m| \frac{D_m c_m}{h^2} + \frac{\varepsilon^2}{\bar\kappa^2} \sum_m w_m \mu_m^2 \frac{D_m^2 c_m}{h^2} \pm \dots\Big) \\ &= \Big(Id + \frac{\varepsilon}{2\bar\kappa} h H_1 + \frac{\varepsilon^2}{3\bar\kappa^2} H_2 + \frac{\varepsilon^3}{4\bar\kappa^3} h H_3 + \dots\Big) \cdot \frac{\kappa}{\bar\kappa} (\frac{\sigma}{\kappa}\varphi + 4\pi B) + R. \end{aligned} \tag{2.6}$$

In R we collect the contributions of boundary terms. Here, odd and even order matrices are defined via

$$H_{2n-1} = -n \sum_{m=1}^{M} w_m |\mu_m|^{2n-1} \frac{D_m^{2n-1}}{h^{2n}}, \quad H_{2n} = \frac{2n+1}{2} \sum_{m=1}^{M} w_m \mu_m^{2n} \frac{D_m^{2n}}{h^{2n}},$$

(the detailed form can be found in appendix A).

Remark 3 *It may be observed that H_2, H_4, etc. are $O(h^2)$ consistent approximations to the second, fourth and higher even space derivatives, respectively. However, since they are based on onesided differences, their stencil is twice as large as corresponding stencils of standard finite difference approximations. Moreover, it can be verified that also the matrices H_1, H_3, etc. are second order approximations to even space derivatives of increasing order. In this form the expansion for the flux strongly resembles the continuous flux equation. However, in the discrete case the odd order sums do not vanish as the continuous integrals do. Instead there remain terms of order h; but these terms can be made arbitrarily small by choosing the grid size appropriately.*

We proceed by inverting the operator on the right using Neumann's series once more. This, in turn, is justified by the following argument. In the maximum norm we have $\|D_m\| = 2$, and hence, by choosing ε such that $q = \frac{2\varepsilon}{\bar\kappa h} < 1$, we can show the uniform boundedness with respect to ε of the matrix series:

$$
\begin{aligned}
\|\sum_{n=1}^{\infty}(-1)^n\frac{n+1}{2}\sum_m w_m\big(\frac{\varepsilon|\mu_m|}{\bar\kappa h}\big)^n D_m^n\| &\le \sum_{n=1}^{\infty}\frac{n+1}{2}\sum_m w_m\big(\frac{\varepsilon|\mu_m|}{\bar\kappa h}\big)^n\|D_m^n\|^n \\
&= \sum_{n=1}^{\infty}\frac{n+1}{2}\big(\frac{2\varepsilon}{\bar\kappa h}\big)^n\sum_m w_m|\mu_m|^n \\
&\le \frac{1}{q}\sum_{n=0}^{\infty} nq^n < \infty
\end{aligned}
$$

Then the Neumann series

$$
\begin{aligned}
\frac{\kappa}{\bar\kappa}\big(\frac{\sigma}{\kappa}\varphi+4\pi B\big) = \Big[Id &- \Big(\frac{\varepsilon}{2\bar\kappa}hH_1+\frac{\varepsilon^2}{3\bar\kappa^2}H_2+\dots\Big) \\
&+ \Big(\frac{\varepsilon}{2\bar\kappa}hH_1+\frac{\varepsilon^2}{3\bar\kappa^2}H_2+\dots\Big)^2 \\
&- \Big(\frac{\varepsilon}{2\bar\kappa}hH_1+\frac{\varepsilon^2}{3\bar\kappa^2}H_2+\dots\Big)^3 \pm \dots\Big](\varphi-R)
\end{aligned}
$$

eventually yields the sought-for asymptotic expansion

$$
\frac{\kappa}{\bar\kappa}4\pi B = \Big[\frac{\kappa}{\bar\kappa}-\frac{\varepsilon}{2\bar\kappa}hH_1+\frac{\varepsilon^2}{\bar\kappa^2}\Big(-\frac{1}{3}H_2+\frac{h^2}{4}H_1^2\Big)+\dots\Big]\varphi+\tilde R. \tag{2.7}
$$

2.1 The discrete SP_1 approximation

When we keep only terms up to second order in ε, the asymptotic expansion (2.7) suggests the following approximation to the radiative transfer equations

$$
\frac{\kappa}{\bar\kappa}4\pi B = \frac{\kappa}{\bar\kappa}\varphi-\frac{\varepsilon^2}{3\bar\kappa^2}H_2\varphi+\Big[-\frac{\varepsilon h}{2\bar\kappa}H_1\varphi+\frac{\varepsilon^2h^2}{4\bar\kappa^2}H_1^2\varphi\Big]+O(\varepsilon^3). \tag{2.8}
$$

This system is no standard discretization of the continuous SP_1 equation, but it is nevertheless consistent up to second order in h with this PDE. For points x_k in the interior of the domain which are sufficiently far from the boundary such that perturbations owing to boundary values can be neglected, we have

$$
(H_2\varphi)_k = \partial_x^2\varphi_k+\frac{7}{12}h^2\partial_x^4\varphi_k+O(h^4)
$$

and, furthermore,

$$
\begin{aligned}
(H_1\varphi)_k &= \partial_x^2\varphi_k+\frac{1}{12}h^2\partial_x^4\varphi_k+O(h^4) \\
(H_1^2\varphi)_k &= \partial_x^4\varphi_k+\frac{1}{6}h^2\partial_x^6\varphi_k+O(h^4),
\end{aligned}
$$

(see appendix A) such that the approximation above is consistent to

$$
\kappa 4\pi B_k = \kappa\varphi_k-\frac{\varepsilon^2}{3(\sigma+\kappa)}\partial_x^2\varphi_k+O(\varepsilon h)+O(\varepsilon^3). \tag{2.9}
$$

Remark 4 *Up to additional terms of order εh this limit equation coincides with the SP_1 equation. Since the underlying asymptotic expansion is only valid when $\varepsilon \le Ch$, the discrepancy is at most of order h^2, which is comparable with the consistency error of standard finite difference approximations of the equation using e.g. central differences.*

2.2 The discrete SP_2 approximation

For the SP_2 approximation we consider terms up to fourth order in expansion (2.7)

$$\begin{aligned}\frac{\kappa}{\bar\kappa}4\pi B =& \frac{\kappa}{\bar\kappa}\varphi - \frac{\varepsilon^2}{3\bar\kappa^2}H_2\varphi - \frac{\varepsilon^4}{5\bar\kappa^4}H_4\varphi + \frac{\varepsilon^4}{9\bar\kappa^4}H_2^2\varphi \\ &+ \Big[-\frac{\varepsilon h}{2\bar\kappa}H_1\varphi + \frac{\varepsilon^2 h^2}{4\bar\kappa^2}H_1^2\varphi - \frac{\varepsilon^3 h}{4\bar\kappa^3}H_3\varphi + \frac{\varepsilon^3 h}{3\bar\kappa^3}H_1H_2\varphi \\ &- \frac{\varepsilon^3 h^3}{8\bar\kappa^3}H_1^3\varphi + \frac{\varepsilon^4 h^2}{4\bar\kappa^4}H_1H_3\varphi + \frac{\varepsilon^4 h^4}{16\bar\kappa^4}H_1^4\varphi\Big] + O(\varepsilon^5).\end{aligned} \tag{2.10}$$

Since $H_4\varphi = H_2^2\varphi + O(h^2)$, we replace H_4 in the equation above. Furthermore, we can use (2.8) to express $H_2^2\varphi$ in terms of $H_2\varphi$

$$\begin{aligned}\frac{\varepsilon^2}{3\bar\kappa^2}H_2^2\varphi &= H_2\Big(\frac{\varepsilon^2}{3\bar\kappa^2}H_2\varphi\Big) \\ &= H_2\Big(\frac{\kappa}{\bar\kappa}(\varphi - 4\pi B) - \frac{\varepsilon h}{2\bar\kappa}H_1\varphi + \frac{\varepsilon^2 h^2}{4\bar\kappa^2}H_1^2\varphi + O(\varepsilon^3)\Big),\end{aligned}$$

which leads to the equation

$$\begin{aligned}\frac{\kappa}{\bar\kappa}4\pi B =& \frac{\kappa}{\bar\kappa}\varphi - \frac{\varepsilon^2}{3\bar\kappa^2}H_2\big(\varphi + \frac{4\kappa}{5\bar\kappa}(\varphi - 4\pi B)\big) \\ &+ \Big[-\frac{\varepsilon h}{2\bar\kappa}H_1\varphi + \frac{\varepsilon^2 h^2}{4\bar\kappa^2}H_1^2\varphi - \frac{\varepsilon^3 h}{4\bar\kappa^3}H_3\varphi + \frac{\varepsilon^3 h}{6\bar\kappa^3}H_1H_2\varphi \\ &- \frac{\varepsilon^3 h^3}{8\bar\kappa^3}H_1^3\varphi + \frac{\varepsilon^4 h^2}{4\bar\kappa^4}H_1H_3\varphi + \frac{\varepsilon^4 h^4}{12\bar\kappa^4}H_1^2H_2\varphi\Big] + O(\varepsilon^5).\end{aligned}$$

This expression can be rewritten by introducing the variable $\xi = \varphi + \frac{4\kappa}{5\bar\kappa}(\varphi - 4\pi B)$:

$$\begin{aligned}\kappa 4\pi B =& \kappa\xi - \varepsilon^2\frac{5\bar\kappa + 4\kappa}{15\bar\kappa}H_2\xi \\ &+ \varepsilon h\Big[-\frac{1}{2}H_1 + \frac{\varepsilon h}{4\bar\kappa}H_1^2 - \frac{\varepsilon^2}{4\bar\kappa^2}H_3 \\ &+ \frac{\varepsilon^2}{3\bar\kappa^2}H_1H_2 - \frac{\varepsilon^2 h^2}{8\bar\kappa^2}H_1^3 + \frac{\varepsilon^3 h}{4\bar\kappa^3}H_1H_3 + \frac{\varepsilon^3 h^3}{16\bar\kappa^3}H_1^4\Big] \\ &\Big(\frac{5\bar\kappa}{5\bar\kappa + 4\kappa}\xi + \frac{4\kappa}{5\bar\kappa + 4\kappa}4\pi B\Big) + O(\varepsilon^5).\end{aligned} \tag{2.11}$$

Remark 5 *Up to $O(\varepsilon h)$ this is a discretization of the continuous SP_2 equation. As in the SP_1 case above, since the asymptotics is only valid when $\varepsilon \le Ch$, the discrete system is second order consistent.*

2.3 The discrete SP_3 approximation

The SP_3 approximation is derived from expansion (2.7) by skipping terms of order seven.

$$
\begin{aligned}
\frac{\kappa}{\bar\kappa}4\pi B =& \frac{\kappa}{\bar\kappa}\varphi - \frac{\varepsilon^2}{3\bar\kappa^2}H_2\varphi - \frac{\varepsilon^4}{5\bar\kappa^4}H_4\varphi + \frac{\varepsilon^4}{9\bar\kappa^4}H_2^2\varphi \\
& - \frac{\varepsilon^6}{7\bar\kappa^4}H_6\varphi + \frac{2\varepsilon^6}{15\bar\kappa^6}H_2H_4\varphi - \frac{\varepsilon^6}{27\bar\kappa^4}H_2^3\varphi \\
& + \Big[-\frac{\varepsilon h}{2\bar\kappa}H_1\varphi - \frac{\varepsilon^3 h}{4\bar\kappa^3}H_3\varphi - \frac{\varepsilon^5 h}{6\bar\kappa}H_5\varphi \\
& + \frac{\varepsilon^2 h^2}{4\bar\kappa^2}H_1^2\varphi + \frac{\varepsilon^3 h}{3\bar\kappa^3}H_1H_2\varphi + \frac{\varepsilon^4 h^2}{4\bar\kappa^4}H_1H_3\varphi + \frac{\varepsilon^5 h}{5\bar\kappa^5}H_1H_4\varphi \\
& + \frac{\varepsilon^5 h}{6\bar\kappa^5}H_2H_3\varphi + \frac{\varepsilon^6 h^2}{6\bar\kappa^6}H_1H_5\varphi + \frac{\varepsilon^6 h^2}{16\bar\kappa^6}H_3^2\varphi \\
& - \frac{\varepsilon^3 h^3}{8\bar\kappa^3}H_1^3\varphi - \frac{\varepsilon^4 h^2}{4\bar\kappa^4}H_1^2H_2\varphi - \frac{\varepsilon^5 h}{6\bar\kappa^5}H_1H_2^2\varphi - \frac{3\varepsilon^6 h^2}{20\bar\kappa^6}H_1^2H_4\varphi \\
& - \frac{\varepsilon^6 h^2}{4\bar\kappa^6}H_1H_2H_3\varphi - \frac{3\varepsilon^5 h^2}{16\bar\kappa}H_1^2H_3\varphi \\
& + \frac{\varepsilon^4 h^4}{16\bar\kappa^4}H_1^4\varphi + \frac{\varepsilon^5 h^3}{6\bar\kappa^5}H_1^3H_2\varphi + \frac{\varepsilon^6 h^2}{6\bar\kappa^6}H_1^2H_2^2\varphi + \frac{\varepsilon^6 h^3}{8\bar\kappa^6}H_1^3H_3\varphi \\
& - \frac{\varepsilon^5 h^5}{32\bar\kappa^5}H_1^5\varphi + \frac{\varepsilon^6 h^6}{64\bar\kappa^6}H_1^6\varphi\Big] + O(\varepsilon^7).
\end{aligned} \tag{2.12}
$$

Using the relations $H_4\varphi = H_2^2\varphi + O(h^2)$ and $H_6\varphi = H_2^3\varphi + O(h^2)$, which are valid in the grid interior, we can rewrite the equation

$$
\begin{aligned}
\frac{\kappa}{\bar\kappa}4\pi B =& \frac{\kappa}{\bar\kappa}\varphi - \frac{\varepsilon^2}{3\bar\kappa^2}H_2\varphi - \frac{4\varepsilon^4}{45\bar\kappa^4}H_2^2\varphi - \frac{44\varepsilon^6}{945\bar\kappa^6}H_2^3\varphi + O(\varepsilon h) + O(\varepsilon^7) \\
=& \frac{\kappa}{\bar\kappa}\varphi - \frac{\varepsilon^2}{3\bar\kappa^2}H_2\Big[Id + \frac{4\varepsilon^2}{15\bar\kappa^2}H_2 + \frac{44\varepsilon^4}{315\bar\kappa^4}H_2^2\Big]\varphi + O(\varepsilon h) + O(\varepsilon^7) \\
=& \frac{\kappa}{\bar\kappa}\varphi - \frac{\varepsilon^2}{3\bar\kappa^2}H_2\Big[Id + 2\Big(Id + \frac{11\varepsilon^2}{21\bar\kappa^2}H_2^2\Big)\frac{2\varepsilon^2}{15\bar\kappa^2}H_2\Big]\varphi + O(\varepsilon h) + O(\varepsilon^7) \\
=& \frac{\kappa}{\bar\kappa}\varphi - \frac{\varepsilon^2}{3\bar\kappa^2}H_2\Big[Id + 2\Big(Id - \frac{11\varepsilon^2}{21\bar\kappa^2}H_2^2\Big)^{-1}\frac{2\varepsilon^2}{15\bar\kappa^2}H_2\Big]\varphi + O(\varepsilon h) + O(\varepsilon^7) \\
=& \frac{\kappa}{\bar\kappa}\varphi - \frac{\varepsilon^2}{3\bar\kappa^2}H_2(\varphi + 2\varphi_2) + O(\varepsilon h) + O(\varepsilon^7).
\end{aligned} \tag{2.13a}
$$

Here, the variable φ_2 is defined via an additional equation

$$
-\frac{11\varepsilon^2}{21\bar\kappa^2}H_2^2\varphi_2 + \varphi_2 = \frac{2\varepsilon^2}{15\bar\kappa^2}H_2\varphi = \frac{2}{5}\Big(\frac{\varepsilon^2}{3\bar\kappa^2}H_2\varphi\Big).
$$

From the expansion above we may also conclude that the right side of the auxiliary equation can be modified according to

$$
\frac{\varepsilon^2}{3\bar\kappa^2}H_2\varphi = \frac{\kappa}{\bar\kappa}(\varphi - 4\pi B) - \frac{2\varepsilon^2}{3\bar\kappa^2}H_2\varphi_2 + O(\varepsilon h),
$$

and therefore the equation is equivalent to

$$
-\frac{9\varepsilon^2}{35\bar\kappa^2}H_2^2\varphi_2 + \varphi_2 = \frac{2}{5\bar\kappa}(\kappa\varphi - 4\pi\kappa B) + O(\varepsilon h). \tag{2.13b}
$$

Remark 6 *Equations (2.13a) and (2.13b) can be regarded as discretizations of the continuous SP_3 equations up to terms of $O(\varepsilon h)$ in the interior of the domain. As before, they are second order consistent with respect to the grid size h.*

3 SP_N approximations for the Adjoint Radiative Transfer Equation

In this section we follow the approach *first optimize, then discretize* and discuss the remaining link, namely, deriving formally the SP_N approximations for the adjoint radiative transfer equation in $\mathbb{R}^3$. As discussed in Section 1 and [5], the adjoint equations for the (continuous) optimal control problem (1.1) can be obtained by equation (1.5).

In order to apply the SP_N approximations to (1.5) we consider an optically thick medium, where the opacity κ is large and therefore the radiation is conveyed in a diffusive way. Hence, we consider the rescaled equation for small value of $\varepsilon = 1/((\kappa_{ref} + \sigma_{ref})\mathbf{x}_{ref})$:

$$-\varepsilon\omega\nabla \mathrm{J} + (\sigma+\kappa)\mathrm{J} = (\sigma+\kappa)G, \tag{3.1}$$

for some function

$$G(\mathbf{x}) = \frac{\alpha_1}{\sigma+\kappa}(\varphi(\mathbf{x}) - \bar{\varphi}(\mathbf{x})) + \frac{\sigma}{4\pi(\sigma+\kappa)}\int_{S^2} \mathrm{J} d\omega. \tag{3.2}$$

As for the radiative transfer equation, Neumann's series is applied to formally compute the inverse of the transport operator:

$$\left(1 - \frac{\varepsilon}{\sigma+\kappa}\omega\cdot\nabla\right)^{-1} = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{(\sigma+\kappa)^n}(\omega\cdot\nabla)^n. \tag{3.3}$$

Now, integrating (3.1) over ω and using (3.3) yields

$$\psi := \int_{S^3} \mathrm{J} d\omega = 4\pi\left(1 + \frac{\varepsilon^2}{3(\sigma+\kappa)^2}\nabla^2 + \frac{\varepsilon^4}{5(\sigma+\kappa)^4}\nabla^4 + \frac{\varepsilon^6}{7(\sigma+\kappa)^6}\nabla^6\right) G + O(\varepsilon^8) \tag{3.4}$$

This yields the following asymptotic expansion for the variable ψ,

$$\left(1 - \frac{\varepsilon^2}{3(\sigma+\kappa)^2}\nabla^2 - \frac{4\varepsilon^4}{45(\sigma+\kappa)^4}\nabla^4 - \frac{44\varepsilon^6}{945(\sigma+\kappa)^6}\nabla^6\right)\psi + O(\varepsilon^8), \tag{3.5}$$

and finally the SP_N, $N = 1, 2, 3$ by neglecting terms of order $O(\varepsilon^4), O(\varepsilon^6)$ and $O(\varepsilon^8)$, respectively.

Summarizing, the SP_1 approximation for the adjoint equation (3.1) in the case of a optically thick medium is given by

$$\psi - \frac{\varepsilon^2}{3(\sigma+\kappa)^2}\nabla^2\psi = \frac{4\pi}{\sigma+\kappa}\alpha_1\left(\varphi(\mathbf{x}) - \bar{\varphi}(\mathbf{x})\right) + \frac{\sigma}{\sigma+\kappa}\psi, \tag{3.6}$$

or equivalently

$$\kappa\psi - \frac{\varepsilon^2}{3(\sigma+\kappa)}\nabla^2\psi = 4\pi\alpha_1\left(\varphi(\mathbf{x}) - \bar{\varphi}(\mathbf{x})\right), \tag{3.7}$$

for the mean intensity $\psi := \int \mathrm{J} d\omega$. Similarly, the SP_2 and SP_3 approximations are given by

$$\kappa\xi - \varepsilon^2 \frac{5\sigma + 9\kappa}{15(\sigma+\kappa)} \nabla^2 \xi = 4\pi\alpha_1 \left(\varphi(\mathbf{x}) - \bar{\varphi}(\mathbf{x})\right), \tag{3.8}$$

where $\xi = \psi + \frac{4\kappa}{5(\sigma+\kappa)} \left(\psi - 4\pi\frac{\alpha_1}{\kappa}\left(\varphi(\mathbf{x}) - \bar{\varphi}(\mathbf{x})\right)\right)$, and

$$\kappa\psi - \frac{\varepsilon^2}{3(\sigma+\kappa)} \nabla^2(\psi + 2\psi_2) = 4\pi\alpha_1 \left(\varphi(\mathbf{x}) - \bar{\varphi}(\mathbf{x})\right), \tag{3.9}$$

$$(\sigma+\kappa)\psi_2 - \frac{9\varepsilon^2}{35(\sigma+\kappa)} \nabla^2\psi_2 - \frac{2}{5}\kappa\psi = -\frac{2}{5}\, 4\pi\alpha_1 \left(\varphi(\mathbf{x}) - \bar{\varphi}(\mathbf{x})\right), \tag{3.10}$$

respectively.

Moreover, for any SP_N approximation, the optimality condition (1.6) reads,

$$\alpha_2 \left(Q(\mathbf{x}) - \bar{Q}(\mathbf{x})\right) + \psi = 0, \tag{3.11}$$

where ψ is the mean intensity given by the SP_N approximations.

Remark 7 *Similarly to [14], let us now consider the optimal control problem subject to the (forward) SP_1 approximation of the radiative transfer equation, i.e.,*

$$\min \mathcal{F}_{SP}(\phi, q) \text{ subject to } \kappa\phi - \frac{\varepsilon^2}{3(\sigma+\kappa)} \nabla^2\phi = q(\mathbf{x}). \tag{3.12}$$

Then, the first-order optimality conditions for (3.12) for $\mathcal{F}_{SP}$,

$$\mathcal{F}_{SP}(\phi, q) = \mathcal{F}(\phi, q/4\pi), \tag{3.13}$$

state ϕ, control q and adjoint variable λ are

$$\begin{aligned} \kappa\phi - \frac{\varepsilon^2}{3(\sigma+\kappa)} \nabla^2\phi &= q(\mathbf{x}), \\ \kappa\lambda - \frac{\varepsilon^2}{3(\sigma+\kappa)} \nabla^2\lambda &= \alpha_1 \left(\phi - \bar{\phi}\right), \\ \alpha_2(q - \bar{q}) + \lambda &= 0. \end{aligned}$$

Obviously, the adjoint and gradient equation of above coincide with (3.11) and (3.6) for the rescaled control $4\pi Q = q$ and Lagrange multiplier $\psi = \lambda$. Moreover, a similar discussion holds true for the SP_2 and SP_3 equations. For theoretical results on the optimality system we refer to [14].

The previous remark and the previous discussion show that – figuratively speaking – deriving the adjoint equations *commutes* with constructing the SP_N approximations.

4 Conclusions

The overall performance of optimization algorithms in radiative transfer relies crucially on the number of degrees of freedom, which is general large due to the angular and frequency dependence of the radiative intensity. Here, we used asymptotic methods in the diffusive regime to reduce the discrete phase space. Future work will focus on moment methods yielding approximate optimization problems via maximum entropy closures.

Acknowledgments

This work has been supported by the Kaiserslautern Excellence Cluster *Dependable Adaptive Systems and Mathematical Modeling*, by the DFG via SFB 568 and project PI 408/3-1, as well as by the DAAD program *PPP Kanada*.

A Matrices used in the expansions

Odd and even order matrices in the asymptotic expansions are defined according to equation (2.6)

$$\begin{aligned} H_{2n-1} &= -n \sum_{m=1}^{M} w_m |\mu_m|^{2n-1} \frac{D_m^{2n-1}}{h^{2n}} \\ &= -n \sum_{\mu_m>0} w_m |\mu_m|^{2n-1} \frac{D_+^{2n-1}}{h^{2n}} - n \sum_{\mu_m<0} w_m |\mu_m|^{2n-1} \frac{D_-^{2n-1}}{h^{2n}}, \end{aligned}$$

and

$$\begin{aligned} H_{2n} &= \frac{2n+1}{2} \sum_{m=1}^{M} w_m \mu_m^{2n} \frac{D_m^{2n}}{h^{2n}} \\ &= \frac{2n+1}{2} \sum_{\mu_m>0} w_m \mu_m^{2n} \frac{D_+^{2n}}{h^{2n}} + \frac{2n+1}{2} \sum_{\mu_m<0} w_m \mu_m^{2n} \frac{D_-^{2n}}{h^{2n}} \end{aligned}$$

Here, we assume for simplicity that we use a quadrature which is exact up to sufficiently high order not only on the whole interval $[-1, 1]$ but also on the positive and negative subinterval. This may be achieved e.g. by using double Gaussian quadrature, which is known to be optimal the realm of radiative transport in slab geometry [22]. But also even order Gauss quadrature, where points are symmetric around 0, has this property. When using a quadrature that does not fulfill the assumption we have to take additional terms due to quadrature errors into account.

Then the first matrix can be immediately written down

$$H_1 = \frac{1}{h^2} \begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{bmatrix},$$

and for points x_k, $k = 1, ..., K-1$ we find the standard central difference discretization for the second derivative

$$(H_1\varphi)_k = \frac{1}{h^2}(\varphi_{k-1} - 2\varphi_k + \varphi_{k+1}) = \partial_x^2 \varphi_k + \frac{1}{12} h^2 \partial_x^4 \varphi_k + O(h^4). \quad \text{(A.1)}$$

The matrix H_1^2 therefore should correspond to the fourth derivative, and in fact

$$H_1^2 = \frac{1}{h^4}\begin{bmatrix} 2 & -3 & 1 & & & & & \\ -3 & 6 & -4 & 1 & & & & \\ 1 & -4 & 6 & -4 & 1 & & & \\ & \ddots & & & & \ddots & & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 6 & -3 \\ & & & & 1 & -3 & 2 \end{bmatrix},$$

such that for $k = 2, ..., K-2$

$$\begin{aligned} (H_1^2\varphi)_k &= \frac{1}{h^4}(\varphi_{k-2} - 4\varphi_{k-1} + 6\varphi_k + -4\varphi_{k+1} + \varphi_{k+2}) \\ &= \partial_x^4\varphi_k + \frac{1}{6}h^2\partial_x^6\varphi_k + O(h^4). \end{aligned} \tag{A.2}$$

The second matrix is the five-band matrix

$$H_2 = \frac{1}{2h^2}\begin{bmatrix} 1 & -2 & 1 & & & & & \\ -1 & 2 & -2 & 1 & & & & \\ 1 & -2 & 2 & -2 & 1 & & & \\ & \ddots & & & & \ddots & & \\ & & 1 & -2 & 2 & -2 & 1 \\ & & & 1 & -2 & 2 & -1 \\ & & & & 1 & -2 & 1 \end{bmatrix},$$

which represents a non-standard five-point stencil discretization of the second derivative at interior points with $k = 2, ..., K-2$

$$\begin{aligned} (H_2\varphi)_k &= \frac{1}{2h^2}(\varphi_{k-2} - 2\varphi_{k-1} + 2\varphi_k + -2\varphi_{k+1} + \varphi_{k+2}) \\ &= \partial_x^2\varphi_k + \frac{7}{12}h^2\partial_x^4\varphi_k + O(h^4). \end{aligned} \tag{A.3}$$

For the SP_2 approximation we compare H_4 and H_2^2. We have

$$H_4 = \frac{1}{2h^4}\begin{bmatrix} * & & & & & & & & & & \\ & & & & & & & & & & \\ 1 & -4 & 6 & -4 & 2 & -4 & 6 & -4 & 1 & & \\ & \ddots & & & & \ddots & & & & \ddots & \\ & & 1 & -4 & 6 & -4 & 2 & -4 & 6 & -4 & 1 \\ & & & & & & & & & & \\ & & & & & & & & & & * \end{bmatrix},$$

and

$$H_2^2 = \frac{1}{4h^4}\begin{bmatrix} * & & & & & & & & & & \\ & & & & & & & & & & \\ 1 & -4 & 8 & -12 & 14 & -12 & 8 & -4 & 1 & & \\ & \ddots & & & & \ddots & & & & \ddots & \\ & & 1 & -4 & 8 & -12 & 14 & -12 & 8 & -4 & 1 \\ & & & & & & & & & & \\ & & & & & & & & & & * \end{bmatrix},$$

such that for $k = 4, ..., K-4$

$$(H_4\varphi)_k = \partial_x^4 \varphi_k + \frac{37}{18} h^2 \partial_x^6 \varphi_k + O(h^4), \text{ and} \tag{A.4}$$

$$(H_2^2\varphi)_k = \partial_x^4 \varphi_k + \frac{7}{6} h^2 \partial_x^6 \varphi_k + O(h^4). \tag{A.5}$$

The matrices H_6 and H_2^3 have to be additionally considered to analyse the SP_3 approximation. They have the form

$$H_6 = \frac{1}{2h^6} \begin{bmatrix} * & & & & & & & & & & & & & & \\ 1 & -6 & 15 & -20 & 15 & -6 & 2 & -6 & 15 & -20 & 15 & -6 & 1 & & \\ & \ddots & & & & & & \ddots & & & & & & \ddots & \\ & & 1 & -6 & 15 & -20 & 15 & -6 & 2 & -6 & 15 & -20 & 15 & -6 & 1 \\ & & & & & & & & & & & & & & * \end{bmatrix},$$

and

$$H_2^3 = \frac{1}{8h^6} \begin{bmatrix} * & & & & & & & & & & & & & & \\ 1 & -6 & 18 & -38 & 63 & -84 & 92 & -84 & 63 & -38 & 18 & -6 & 1 & & \\ & \ddots & & & & & & \ddots & & & & & & \ddots & \\ & & 1 & -6 & 18 & -38 & 63 & -84 & 92 & -84 & 63 & -38 & 18 & -6 & 1 \\ & & & & & & & & & & & & & & * \end{bmatrix},$$

such that for $k = 6, ..., K-6$

$$(H_6\varphi)_k = \partial_x^6 \varphi_k + \frac{19}{4} h^2 \partial_x^8 \varphi_k + O(h^4), \text{ and} \tag{A.6}$$

$$(H_2^3\varphi)_k = \partial_x^6 \varphi_k + \frac{7}{4} h^2 \partial_x^8 \varphi_k + O(h^4). \tag{A.7}$$

We finally note that D_+ and D_- almost commute in the sense that $(D_+D_-\varphi)_k = (D_-D_+\varphi)_k$ for all indices except the first and last. This can be carried over to the matrices H_n, which fulfill the corresponding relations $(H_pH_q\varphi)_k = (H_qH_p\varphi)_k$.

References

[1] V. I. Agoshkov, C. Bardos, *Optimal Control Approach in Inverse Radiative Heat Transfer Problems: The Problem of the Boundary Function*, **5**, ESAIM: Control Optimisation and Calculus of Variations, pp. 259-278, (2000)

[2] J. M. Banoczi, C. T. Kelley, *A Fast Multilevel Algorithm for the Solution of Nonlinear Systems of Conductive-Radiative Heat Transfer Equations in Two Space Dimensions*, **20**, SIAM J. Sci. Comp., pp. 1214-1228, (1999)

[3] M. K. Choudhary and N. T. Huff. *Mathematical modeling in the glass industry: An overview of status and needs.* Glastech. Ber. Glass Sci. Technol., 70:363-370, 1997.

[4] M. Frank, M. Schäfer and R. Pinnau. *A Hierarchy of Approximations to the Radiative Heat Transfer Equations: Modelling, Analysis and Simulation.* M3AS, 15:643-665, 2006.

[5] M. Herty, R. Pinnau, M. Seaid, *On Optimal Control Problems in Radiative Transfer*, preprint, 2005

[6] M. T. Laitinen and T. Tiihonen, *Integro-differential equation modelling heat transfer in conduction, radiating and semitransparent materials*, Mathematical Methods in Applied Sciences, **21**, pp 375, 1998

[7] C. T. Kelley, *Existence and uniqueness of solutions of nonlinear systems of conductive-radiative heat transfer equations*, Trans. Theory and Statistical Physics, **25**, pp. 249-260, (1996)

[8] E.W. Larsen, G. Thömmes, A. Klar, M. Seaïd, and T. Götz. *Simplified P_n approximations to the equations of radiative heat transfer in glass.* J. Comp. Phys., 183:652-675, 2002.

[9] D. Levermore. *Moment closure hierarchies for kinetic theories.* J. Stat. Phys., 83, 1996.

[10] S. Manservisi, K. Heusermann, *On some optimal control problems for the heat radiative transfer equation*, **5**, ESAIM: Control, Optimisation and Calculus of Variations, pp 425-444, (2000)

[11] D. Mihalas and B. S. Mihalas, *Foundations of Radiation Hydrodynamics.* Oxford University Press, New York, 1983.

[12] M. F. Modest, *Radiative Heat Transfer.* McGraw-Hill, 1993.

[13] S. Pereverzyev, R. Pinnau and N. Siedow. *Regularized Fixed Point Iterations for Nonlinear Inverse Problems.* Inverse Problems 22, 1-22 (2006)

[14] R. Pinnau, *Analysis of Optimal Boundary Control For Radiative Heat Transfer Modelled by the SP_1 System*, preprint 2005.

[15] R. Pinnau, A. Schulze. *Newton's Method for Optimal Temperature-Tracking of Glass Cooling Processes.* to appear in IPSE, 2006.

[16] R. Pinnau and G. Thömmes. *Optimal boundary control of glass cooling processes.* M2AS, 120:1261, 2004.

[17] M. Seaïd and A. Klar, *Efficient Preconditioning of Linear Systems Arising from the Discretization of Radiative Transfer Equation*, Lecture Notes in Computational Science and Engineering, **35**, pp 211-236, 2003

[18] M. Seaïd, M. Frank, A. Klar, R. Pinnau and G. Thömmes, *Efficient Numerical Methods for Radiation in Gas Turbines*, J. Comp. Applied Math., **170**, pp 217-239, 2004

[19] M. Seaïd, A. Klar and R. Pinnau, *Numerical Solvers for Radiation and Conduction in High Temperature Gas Flows*, Flow, Turbulence and Combustion, **3**, pp 413-432, 2005

[20] G. Thömmes, *Radiative Heat Transfer Equations for Glass Cooling Problems: Analysis and Numerics*, PhD Thesis, 2002.

[21] G. Thömmes, R. Pinnau, M. Seaid, T. Götz, and A. Klar. *Numerical methods and optimal control for glass cooling processes.* TTSP, 31(4-6):513-529, 2002.

[22] J. Pitkaränta, L. R. Scott, *Error estimates for the combined spatial and angular approximations of the transport equation for slab geometry*, SIAM J. Numer. Anal., 20(5):922-950, 1983