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A bootstrap test for comparing images in surface inspection

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Abstract: We discuss the detection of defects in woven textiles with semi-regular surface structure treating it as a nonparametric testing problem. We show that the wild bootstrap may be used to approximate the law of the test statistic. We develop an algorithm allowing not only to test for a defect, but also to get some information on its location and shape.

1 Introduction

In industrial quality control, detecting defects on surfaces plays a major role. So far quality control in many cases still is performed manually, by checking for defects using random sampling and applying quality control charts. The results of this manual procedure is of course depending on the examiner. Even, as it is stated in Chetverikov [2], though humans have the ability to easily find imperfections in spatial structures, they suffer from physical fatigueness reducing their performance. Therefore, automatic inspection systems tend to replace or at least augment human eyes by cameras, part of their brains by computers and part of their abilities to detect the error by algorithms and software. Some applications for automatic defect detection are summarized in Kohrt [16].

In this work, we consider in particular defect detection in textures. Numerous methods have been designed to solve particular texture inspection tasks. Cohen et al. [4] used MRF models for defect inspection in textile surface, Chetverikov [3] using regularity and local orientation for finding defects in texture. Meanwhile Sezer et al. [20] use Independent Component Analysis for the same purpose.

We treat defect detection as a hypothesis testing problem with the null hypothesis representing the absence of defects. As a test case for the general approach to the inspection of surfaces, a practical problem from production control is considered: the quality of woven fabrics is monitored during production to maintain the quality standards required by customers. Up to fifty different kinds of defects are known to be present, and an automated system of several cameras and corresponding data-processing hard- and software has to detect and classify defects. The algorithms currently available are not completely satisfactory as, e.g., several tuning parameters have to be chosen in a heuristic fashion during implementation of the system (compare Daul et al. [5]).

The test statistics, which we use for defect detection, have a rather complicated structure. Therefore, we apply the bootstrap for approximating its distribution. Bootstrap techniques replace and, for small or moderate sample sizes, frequently improve the classic asymptotic analysis where the limit distribution for sample size going to ∞ is derived. Typical applications are tests for comparing two random signals or two noisy images with each other. Here, the hypothesis that both signals or images are generated by the same random mechanism is accepted if $T \leq c_\alpha$ for some test statistic T derived from some basic intuition. For choosing the significance bound c_α such that the resulting test has the required level α, the distribution of T under the hypothesis or at least an approximation of it has to be known. c_α depends on α as well as on the sample size and may be calculated approximately either from the limit law of T or by resampling methods like the bootstrap. For simple statistics in simple parametric models based on at least moderately large sample sizes, the classical asymptotic approach works usually well. However, in signal and image analysis, frequently more complicated statistics in complex parametric or even nonparametric models are of interest. In such situations, the limit distribution often provides reasonable approximations to c_α only for sample sizes which are too large for many applications.

We apply the construction of bootstrap tests to the automatic surface inspection problem in industrial quality control discussed above. The task is the detection and classification of certain defects in woven textiles which have a kind of semiregular and additionally noisy surface structure which is hard to reproduce by simple parametric models. Therefore, we adopt a nonparametric view. In a first step, we smooth the surface under consideration locally, and, then, compare the resulting denoised image with another one derived from a similar specimen or with different part of the same specimen which is known to be free of defects. We construct an appropriate test where the significance bound is calculated approximately by the wild bootstrap. We prefer to use this resampling method to the more common residual-based bootstrap as it is simple to implement and works well in a heteroscedastic situation, i.e. the variance of the noise is not constant over the image which is a common feature of surface inspection problems, compare, e.g., Figure 2 of Daul et al. [5].

2 The testing problem

We assume that we have a pair of noisy images $\mathbf{Y}, \tilde{\mathbf{Y}}$ of same size, the first one known to be without defect, the other one to be tested for the presence of a defect. We assume that the images are observed on a regular grid $\mathbf{x}_{ij} = (\frac{i}{n}, \frac{j}{n}), i, j = 0, \ldots, n$, in the unit square

$$\mathbf{Y}_{ij} = \mathbf{m}_I(\mathbf{x}_{ij}) + \varepsilon_{ij}, \qquad \tilde{\mathbf{Y}}_{ij} = \mathbf{m}_{II}(\mathbf{x}_{ij}) + \tilde{\varepsilon}_{ij}, \quad i, j = 0, \ldots, n, \tag{1}$$

where $\varepsilon_{ij}, \tilde{\varepsilon}_{ij}, i, j = 0, \ldots, n$, are independent with mean zero and finite variances $\mathrm{var}(\varepsilon_{ij}) = \mathrm{var}(\tilde{\varepsilon}_{ij}) = \sigma^2(\mathbf{x}_{ij})$ and uniformly bounded fourth moments $\mathrm{E}\ \varepsilon_{ij}^4, \mathrm{E}\ \tilde{\varepsilon}_{ij}^4 \leq C < \infty,\ i, j = 0, \ldots, n$. The random residuals are not only representing the observational noise, but also the random fine-scale structure in the woven textile.

To detect and to localize defects, we compare corresponding rows resp. columns of the two images, i.e. for some given $0 \leq k \leq n$, we consider the k^{th} rows $Y_i = \mathbf{Y}_{ik}, \tilde{Y}_i = \tilde{\mathbf{Y}}_{ik}, i = 0, \ldots, n$, or the two columns $Y_i = \mathbf{Y}_{ki}, \tilde{Y}_i = \tilde{\mathbf{Y}}_{ki}, i = 0, \ldots, n$. In both cases, we end up with the following one-dimensional setup

$$Y_i = m_I(x_i) + \varepsilon_i, \qquad \tilde{Y}_i = m_{II}(x_i) + \tilde{\varepsilon}_i, \qquad x_i = \frac{i}{n}, \quad i = 0, ..., n, \tag{2}$$

where, for, e.g., comparing the k^{th} rows, $m_I(x_i) = \mathbf{m}_I(\mathbf{x}_{ik}), m_{II}(x_i) = \mathbf{m}_{II}(\mathbf{x}_{ik}), i = 0, \ldots, n$, and the $\varepsilon_i = \varepsilon_{ik}, \tilde{\varepsilon}_i = \tilde{\varepsilon}_{ik}, i = 0, \ldots, n$, are independent with mean zero, finite variances $\text{var}(\varepsilon_i) = \text{var}(\tilde{\varepsilon}_i) = \sigma^2(x_i)$ and uniformly bounded fourth moments. First, we discuss the problem of testing if $m_I = m_{II}$.

We do not assume a specific parametric form of the general pattern underlying the observed image, which would be hard to specify in the case of woven textiles. Therefore, m_I, m_{II} may be arbitrary functions except for satisfying some general regularity assumptions. Then, (2) is a setup which has been studied extensively in nonparametric regression analysis, compare, e.g., [6, 10]. As the design points x_i are deterministic and equidistant, we estimate m_I, m_{II} by Priestley-Chao [19] kernel estimates $\widehat{m}_I, \widehat{m}_{II}$ which are local weighted averages of the data:

$$\widehat{m}_I(x,h) = \frac{1}{n+1}\sum_{i=0}^{n} K_h(x - x_i)Y_i, \qquad \widehat{m}_{II}(x,h) = \frac{1}{n+1}\sum_{i=0}^{n} K_h(x - x_i)\tilde{Y}_i,$$

where $K_h(.) = h^{-1}K(./h)$ denotes a kernel function rescaled by the bandwidth h.

Several tests for the equality of functions based on kernel estimates have been studied, in particular for testing if a certain parametric model adequately describes the data. There, the nonparametric kernel fit is compared with a parametric function estimate based on the same sample, compare, e.g., Hall and Hart [9] and Härdle and Mammen [12]. Härdle et.al [13] use the same kind of idea for testing a parametric versus a semiparametric model. We adapt this approach to comparing two nonparametric kernel estimates based on two different samples. We consider the testing problem

$$H_0 : m_I(x_i) = m_{II}(x_i),\ i = 0, ..., n, \text{ against } H_1 : m_I(x_i) \neq m_{II}(x_i) \text{ for some } i.$$

To perform the test, we look at some distance between the function estimates $\widehat{m}_I(x,h)$ and $\widehat{m}_{II}(x,h)$, and we reject the hypothesis H_0 if this distance is too large. Following, Härdle and Mammen [12], we use a standardized L_2-distance between the two estimate

$$T_n = n\sqrt{h}\int \left(\widehat{m}_{II}(x,h) - \widehat{m}_I(x,h)\right)^2 dx \tag{3}$$

To apply the test we have to determine a bound $\tau_{\alpha,n}$ depending on the level α of the test and on sample size n such that, under the hypothesis, $\text{pr}(T_n > \tau_{\alpha,n}) \leq \alpha$. For that purpose, we have to approximate the distribution of T_n under the hypothesis. We first derive an asymptotic approximation by a Gaussian distribution which, for practically purposes, is good enough only for large n - compare, e.g., Franke and Härdle [7] for the related problem of spectrum estimation. We use this intermediate result to show that the bootstrap provides a valid approximation of the distribution of T_n. This allows to calculate a useful bootstrap approximation $\tau^*_{\alpha,n}$ of $\tau_{\alpha,n}$ which is used in our defect detection algorithm.

3 Asymptotic properties of the test statistic

We first state the assumptions on the kernel and bandwidth.

(K1) The kernel K is a symmetric, twice continuously differentiable function with compact support $[-1, 1]$, standardized to $\int K(u)\, du = 1$.

(K2) The bandwidth h fulfills $h = h_n \sim cn^{-1/5}$ for some $c > 0$.

The assumptions on K are standard in nonparametric regression. The assumption on the bandwidth h could be relaxed, but $n^{-1/5}$ is the optimal rate for minimizing the integrated mean-squared error $\int \big(\widehat{m}_I(x,h) - m_I(x)\big)^2 dx$ for $n \to \infty$ and, therefore, is most suitable for our test statistic T_n. For the underlying functions to be estimated and tested and for the variance of the residuals we assume

(A1) $m_I(.), m_{II}(.)$ are twice continuously differentiable.

(A2) m_{II} can be written as $m_{II}(x) = m_I(x) + c_n\Delta_n(x)$ with $c_n = (n\sqrt{h})^{-1/2}$ and $\Delta_n(x)$ bounded uniformly in x and n.

(A3) $\sigma^2(x)$ is bounded away from 0 and from ∞, uniformly in $x \in [0,1]$, and it satisfies a Lipschitz condition.

Assumption (A2) is imposed as we also want to study the asymptotic behaviour of our test if the alternative H_1 holds. If we would keep the distance between m_I, m_{II} fixed, then the test will detect the violation of the hypothesis with probability going to 1. To get an interesting asymptotic behaviour, we have to apply the test to functions with a distance going to 0 for $n \to \infty$. Mark that the hypothesis H_0 corresponds to $\Delta_n(.) \equiv 0$ in (A2).

In the first proposition, we show that the distribution $\mathcal{L}(T_n)$ of T_n may be approximated by a Gaussian distribution, where the distance between these distributions is measured by the following truncated version of the Mallows distance, which is also used by Härdle and Mammen in [12]

$$d_2(\mu,\nu) = \inf_{X,Y:\mathcal{L}(\mathrm{X})=\mu,\mathcal{L}(\mathrm{Y})=\nu} \min(\mathrm{E}\,\|X-Y\|^2, 1)$$

Convergence in this distance is equivalent to weak convergence. In the following, we denote by $f * g(x) = \int f(x-y)g(y)dy$ the convolution of two functions f, g and by $g^{(*k)}(x)$ the k-fold convolution of g with itself.

Proposition 1 *Assume that model (2) holds and (K1), (K2), (A1)-(A3) are satisfied. Then,*

$$d_2\Big(\mathcal{L}(T_n), \mathcal{N}(B_h, \sigma_T^2)\Big) \to 0$$

where asymptotic bias B_h and variance σ_T^2 of T_n are given by

$$B_h = B_h^0 + B_h^1,$$

$$B_h^0 = \frac{2}{\sqrt{h}} \int \sigma^2(x)dx \int K^2(u)du,$$

$$B_h^1 = \int (K_h * \Delta_n(x))^2 dx,$$

$$\sigma_T^2 = 8 \int \sigma^4(x)dx\, K^{(*4)}(0).$$

In particular, $B_h^1 \geq 0$ and, under the hypothesis H_0, $B_h^1 = 0$.

We postpone the proof to the appendix. Mark that the variance of T_n converges to σ_T^2 for $n \to \infty$ whereas the bias diverges as $h \to 0$. In general, the difference B_h^1 between the bias under H_0 and H_1 is of order h^{-1} and, therefore, larger than B_h^0, as, e.g., for $\Delta_n(x) \equiv \delta$ we have $B_h^1 = \delta^2 \int K_h^2(x)\, dx = h^{-1}\delta^2 \int K^2(x)\, dx$.

4 The bootstrap

We could use Proposition 1 directly to approximate the quantile $\tau_{\alpha,n}$ of the distribution of the test statistic T_n under H_0 by replacing the unknown integrals of $\sigma^2(x)$ resp. $\sigma^4(x)$ by consistent estimates. However, this would introduce an additional approximation error where, for moderate sample size, $\mathcal{L}(T_n)$ is already not well approximated by the asymptotic normal law with known parameters B_h^0, σ_T^2. Therefore, we prefer to use a bootstrap approximation of $\mathcal{L}(T_n)$, instead. In the heteroscedastic model which we are considering, the approach of bootstrapping from the sample residuals, which is common in regression (compare section 5.3 of [10]) would require to first estimate the variance function $\sigma^2(x)$ as well. We prefer to use the direct approach of the wild bootstrap proposed by Wu [21] (see also Liu [17], Mammen [18], Härdle and Mammen [12]).

As a first step of the bootstrap, we estimate the residuals by

$$\hat{\varepsilon}_i = Y_i - \widehat{m}_I(x_i, h), \qquad \hat{\tilde{\varepsilon}}_i = \tilde{Y}_i - \widehat{m}_{II}(x_i, h), \quad i = 0, \ldots, n.$$

Centering the sample residuals by their sample mean, we get

$$\hat{\varepsilon}_i^0 = \hat{\varepsilon}_i - \frac{1}{n+1}\sum_{j=0}^{n} \hat{\varepsilon}_j, \qquad \hat{\tilde{\varepsilon}}_i^0 = \hat{\tilde{\varepsilon}}_i - \frac{1}{n+1}\sum_{j=0}^{n} \hat{\tilde{\varepsilon}}_j, \quad i = 0, \ldots, n.$$

Now, we consider the original data and all random variables derived from them as fixed, i.e. we condition on $Y_i, \tilde{Y}_i, i = 0, \ldots, n$. To construct the bootstrap residuals $\varepsilon_i^*, i = 0, \ldots, n$, we consider distributions $\widehat{F}_i$ having first moment 0, second moment $(\hat{\varepsilon}_i^0)^2$ and third moment $(\hat{\varepsilon}_i^0)^3$, i.e. for a random variable Z with distribution $\widehat{F}_i$, we have

$$\mathrm{E}Z = 0, \ \mathrm{E}Z^2 = (\hat{\varepsilon}_i^0)^2, \ \mathrm{E}Z^3 = (\hat{\varepsilon}_i^0)^3. \tag{4}$$

Then, we sample ε_i^* from the distribution $\widehat{F}_i$. Analogously, we get $\tilde{\varepsilon}_i^*$ from a distribution with moments specified by $\hat{\tilde{\varepsilon}}_i^0$.

A simple choice for, e.g., $\widehat{F}_i$ is a two-point distribution

$$\widehat{F}_i = \gamma\delta_a + (1-\gamma)\delta_b,$$

where δ_a, δ_b denote point masses in a, b, i.e. we generate ε_i^* randomly such that

$$\mathrm{pr}^*(\varepsilon_i^* = a) = \gamma, \quad \mathrm{pr}^*(\varepsilon_i^* = b) = 1 - \gamma.$$

pr^* denotes the conditional probability given the data $Y_i, \tilde{Y}_i, i = 0, \ldots, n$. As $\widehat{F}_i$ has to satisfy (4), some elementary calculations show that the parameters a, b, γ at each location x_i are given by

$$a = \frac{1-\sqrt{5}}{2}\,\hat{\varepsilon}_i^0, \quad b = \frac{1+\sqrt{5}}{2}\,\hat{\varepsilon}_i^0, \quad \gamma = \frac{5+\sqrt{5}}{10}$$

Once we have the residuals in the bootstrap world, we construct our bootstrap samples by

$$Y_i^* = \widehat{m}_I(x_i, g) + \varepsilon_i^*, \qquad \tilde{Y}_i^* = \widehat{m}_{II}(x_i, g) + \tilde{\varepsilon}_i^*, \quad i = 0, \ldots, n.$$

The functions to be estimated in the bootstrap world are kernel estimates $\widehat{m}_I(x, g), \widehat{m}_{II}(x, g)$ where, following Franke et al. [8], the reference bandwidth g is chosen such that $h, g \to 0$, $\frac{h}{g} \to$

0 for $n \to \infty$. Therefore, $\widehat{m}_I(x,g)$, e.g., is an oversmooth estimate of $m_I(x)$ which is necessary to reproduce the bias in the bootstrap world correctly - for a discussion of this point, compare Franke and Härdle [7] for the Priestley-Chao kernel estimate in the related context of spectrum estimation.

Using the bootstrap data, we calculate kernel estimates in the bootstrap world

$$\widehat{m}_I^*(x,h) = \frac{1}{n+1}\sum_{i=0}^{n} K_h(x-x_i)Y_i^*, \qquad \widehat{m}_{II}^*(x,h) = \frac{1}{n+1}\sum_{i=0}^{n} K_h(x-x_i)\tilde{Y}_i^*,$$

Mark that, e.g., $\widehat{m}_I^*(x,h)$ estimates $\widehat{m}_I(x,g)$ in the bootstrap world, and analogously for $\widehat{m}_{II}^*(x,h), \widehat{m}_{II}(x,g)$.

For applying the wild bootstrap, it is important that the bootstrap estimation error $\widehat{m}_I^*(x,h) - \widehat{m}_I(x,g)$ approximates the real estimation error $\widehat{m}_I(x,h) - m_I(x)$ well. As, under assumption (K2), both converge to 0 with the rate $(nh)^{-1/2}$, the distance between the distributions of the appropriately rescaled errors should converge to 0. In Theorem 1 of Härdle and Marron [11] it is shown that the difference between the distributions of $\sqrt{nh}\{\widehat{m}_I(x,h) - m_I(x)\}$ and of $\sqrt{nh}\{\widehat{m}_I^*(x,h) - \widehat{m}_I(x,g)\}$, the latter conditional on the original data $Y_0,\ldots,Y_n$, vanishes for $n \to \infty$ provided that $h = cn^{-1/5}, g \to 0, \frac{h}{g} \to 0$. The remaining technical assumptions, needed for that result, are essentially weaker than (A1)-(A3), (K1)-(K2). Cao-Abad [1] extends those results and provides rates of convergence. Härdle and Marron [11] also discuss the problem of choosing the reference bandwidth g in more detail. Mark that Härdle and Marron discuss the Nadaraya-Watson kernel estimate, but, for equidistant design, the difference to the Priestley-Chao estimate is asymptotically neglible.

Now, the bootstrap test statistic can be constructed as follows. Analogously to (3) we define

$$T_n^* = nh^{1/2}\int \left(\widehat{m}_I^*(x,h) - \widehat{m}_{II}^*(x,h)\right)^2 dx$$

The distribution of T_n^* is known, given the data. Therefore, we can use Monte Carlo simulation to approximate a $(1-\alpha)$-quantile $\tau_{\alpha,n}^*$ of $\mathcal{L}^*(T_n^*)$. Using this bootstrap quantile, we finally reject the hypothesis $H_0 : m_I = m_{II}$ if $T_n > \tau_{\alpha,n}^*$.

The validity of this bootstrap based test is given by the following result, which is analogous to Theorem 2 of Härdle and Mammen [12]. The proof is again postponed to the appendix. We need an additional assumption on the residuals

(E1) E $\exp(t\varepsilon_i)$, E $\exp(t\tilde{\varepsilon}_i)$ are uniformly bounded in $i = 1, 2, \ldots$ for all $|t|$ small enough.

Theorem 1 *Assume (A1)-(A3), (E1), (K1), (K2). Let $g \to 0$ such that $h/g \to 0$. Then*

$$d_2(\mathcal{L}^*(T_n^*), \mathcal{L}(T_n)) \to 0$$

in probability.

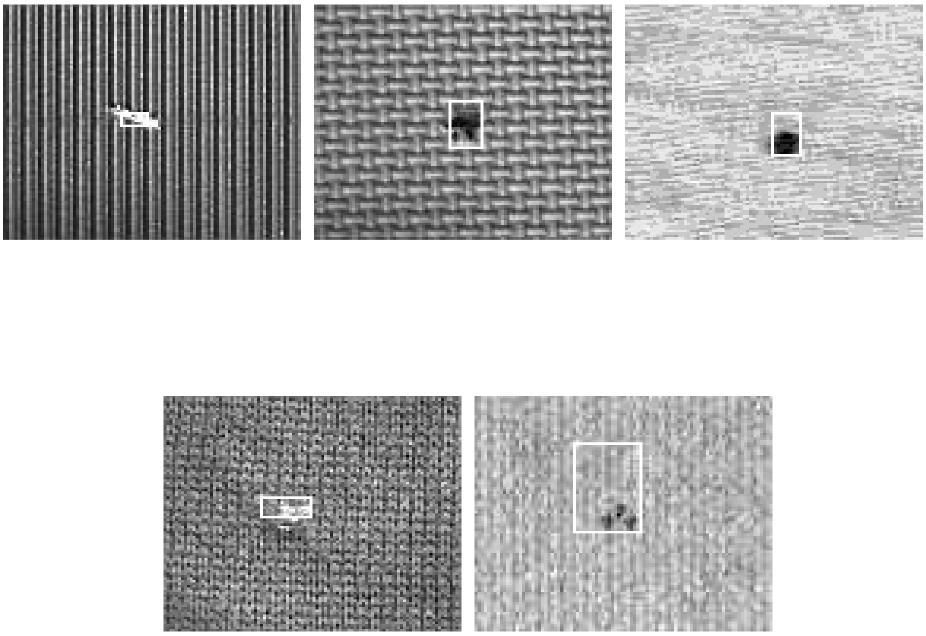

Figure 1: Some examples of defect detection

5 An algorithm for defect detection and localization

We consider two images $\mathbf{Y}, \tilde{\mathbf{Y}}$ as in (1), the first one is known to be free of defects. To detect and localize potential defects in the second one, we compare both images rowwise and columnwise. We combine the results to vectors $\mathbf{c} \in \{0,1\}^N, \mathbf{r} \in \{0,1\}^M$.

As short-hand notation, we use $\mathbf{Y}_{\cdot j}, \tilde{\mathbf{Y}}_{\cdot j}$ respectively for the j^{th} column of $\mathbf{Y}, \tilde{\mathbf{Y}}$. Now, for any $j = 0, ..., n$, we model $\mathbf{Y}_{\cdot j}$ and $\tilde{\mathbf{Y}}_{\cdot j}$ as in (2), and we test for each j if $H_0 : m_I = m_{II}$ holds or not using the bootstrap based test of Section 4. If the hypothesis is rejected, we set $\mathbf{c}_j = 1$, and $\mathbf{c}_j = 0$ else.

Analogously, we compare the two rows $\mathbf{Y}_{i\cdot}, \tilde{\mathbf{Y}}_{i\cdot}$, and we set $\mathbf{r}_i = 1$ if $H_0 : m_I = m_{II}$ is rejected, and $\mathbf{r}_i = 0$ else. Let

$$i_{min} = \min\{i; \mathbf{r}_i = 1\}, i_{max} = \max\{i; \mathbf{r}_i = 1\}, j_{min} = \min\{j; \mathbf{c}_j = 1\}, j_{max} = \max\{j; \mathbf{c}_j = 1\}.$$

Then the rectangle

$$\big[(i_{min}, j_{min}), (i_{max}, j_{max})\big] = \{(i,j);\ i_{min} \le i \le i_{max}, j_{min} \le j \le j_{max}\}$$

is detected as the defect area of the image. Of course, the algorithm can be straightforwardly modified to cover also the case of non-quadratic images.

Figure 1 shows some examples of areas of defect detected by the algorithm in real textures. In all cases, the image size was 102×96 pixels. We have used as a bandwidth $h \approx 0.4$ close to the respective optimal bandwidth $n^{-1/5}$, and as a preliminary bandwidth for constructing the bootstrap data $g = 2h$. Using the bootstrap, we approximated the $\tau^*_{\alpha,n}$ for level $\alpha = 0.05$ by Monte Carlo simulation based on 500 artificially generated samples of bootstrap data. Before applying the test, we removed linear trends in the column resp. row series data due to illumination (compare Halim [14] for details).

6 The two - dimensional case

If we are only interested in the presence of a defect, not in its location and shape, we could apply directly a two-dimensional version of the bootstrap based test of Section 4 instead of comparing line by line and column by column using the one-dimensional test. The derivation of the corresponding theory is a straightforward exercise involving only more cumbersome notation. Therefore, we only the state the results, but do not give the proofs - compare also Härdle and Mammen [12] who consider also the d-dimensional case in their testing problem.

Now, our data are generated by the model (1). Mark that, here, the sample size is not $n+1$ but $(n+1)^2$; to simplify notation, we set $N = n^2$.

First, let us formulate the modified assumptions. (A1), (A3) and (K1) are not changed at all except for $\mathbf{m}_I, \mathbf{m}_{II}, \sigma^2$ and K being functions of $\mathbf{x} = (x_1, x_2)^T \in \mathbb{R}^2$. (A2), (K2) are replaced by

(A2') $\mathbf{m}_{II}$ can be written as $\mathbf{m}_{II}(\mathbf{x}) = \mathbf{m}_I(\mathbf{x}) + c_N \Delta_N(\mathbf{x})$ with $c_N = (Nh)^{-1/2}$ and $\Delta_N(\mathbf{x})$ bounded uniformly in $\mathbf{x}$ and N.

(K2') The bandwidth h fulfills $h = h_N \sim cN^{-1/6}$.

Remark that (K2') specifies the mean-squared error optimal rate of bandwidth in two dimensions under assumption (A1). Now, our test statistic is

$$T_N = Nh \int\int \big(\widehat{\mathbf{m}}_{II}(\mathbf{x}, h) - \widehat{\mathbf{m}}_I(\mathbf{x}, h)\big)^2 dx_1 dx_2$$

where $\widehat{\mathbf{m}}_I, \widehat{\mathbf{m}}_{II}$ denote the Priestley-Chao estimate based on $\mathbf{Y}_{ij}, \tilde{\mathbf{Y}}_{ij}$ respectively, i.e., of the form

$$\widehat{\mathbf{m}}_I(\mathbf{x}, h) = \frac{1}{N} \sum_{i,j=0}^{n} K_h(\mathbf{x} - \mathbf{x}_{ij}) \mathbf{Y}_{ij}$$

with $K_h(\mathbf{u}) = h^{-2} K(\mathbf{u}/h), \mathbf{u} \in \mathbb{R}^2, h > 0$ and $\mathbf{x}_{ij} = \frac{1}{n}(i, j), i, j = 0, ..., n$. We have

Proposition 2 *Assume (A1),(A2'),(A3), (K1),(K2'). Then,*

$$d_2(\mathcal{L}(T_N), \mathcal{N}(B_h, \sigma_T^2)) \to 0$$

with $B_h = B_h^0 + B_h^1$,

$$B_h^0 = \frac{2}{h} \int\int \sigma^2(\mathbf{x}) dx_1 dx_2 \int\int K^2(\mathbf{u}) du_1 du_2$$

$$B_h^1 = \int\int \Big((K_h * \Delta_N(\mathbf{x}) \Big)^2 dx_1 dx_2$$

$$\sigma_T^2 = 8 \int\int \sigma^4(\mathbf{x}) dx_1 dx_2 \, K^{(*4)}(0)$$

Analogously, the bootstrap result of Theorem 1 can also be generalized to two-dimensions.

Theorem 2 *Assume (A1), (A2'), (A3), (E1), (K1),(K2'). Let $g \to 0$ such that $h/g \to 0$. Then*

$$d_2(\mathcal{L}^*(T_N^*), \mathcal{L}(T_N)) \to 0$$

in probability.

Here, T_N^* is constructed in the following manner, quite analogously to the one-dimensional case. First, we consider the sample residuals and center them:

$$\begin{aligned}
\hat{\varepsilon}_{ij} &= \mathbf{Y}_{ij} - \widehat{\mathbf{m}}_I(\mathbf{x}_{ij}, h), \qquad \hat{\tilde{\varepsilon}}_{ij} = \tilde{\mathbf{Y}}_{ij} - \widehat{\mathbf{m}}_{II}(\mathbf{x}_{ij}, h), \quad i,j = 0,\ldots,n, \\
\hat{\varepsilon}^0_{ij} &= \hat{\varepsilon}_{ij} - \frac{1}{(n+1)^2}\sum_{k,l=0}^{n} \hat{\varepsilon}_{kl}, \qquad \hat{\tilde{\varepsilon}}^0_{ij} = \hat{\tilde{\varepsilon}}_{ij} - \frac{1}{(n+1)^2}\sum_{k,l=0}^{n} \hat{\tilde{\varepsilon}}_{kl}, \quad i,j = 0,\ldots,n.
\end{aligned}$$

We choose distributions $\widehat{F}_{ij}$ having first moment 0, second moment $(\hat{\varepsilon}^0_{ij})^2$ and third moment $(\hat{\varepsilon}^0_{ij})^3$, e.g.

$$\widehat{F}_{ij} = \gamma\delta_a + (1-\gamma)\delta_b, \quad a = \frac{1-\sqrt{5}}{2}\hat{\varepsilon}^0_{ij}, \quad b = \frac{1+\sqrt{5}}{2}\hat{\varepsilon}^0_{ij}, \quad \gamma = \frac{5+\sqrt{5}}{10}.$$

We generate the bootstrap residuals ε^*_{ij} by drawing a random variable from the distribution $\widehat{F}_{ij}$. Analogously, we construct the $\tilde{\varepsilon}^*_{ij}$. Once we have the residuals in the bootstrap world, we construct our bootstrap samples by

$$\mathbf{Y}^*_{ij} = \widehat{\mathbf{m}}_I(\mathbf{x}_{ij}, g) + \varepsilon^*_{ij}, \qquad \tilde{\mathbf{Y}}^*_{ij} = \widehat{\mathbf{m}}_{II}(\mathbf{x}_{ij}, g) + \tilde{\varepsilon}^*_{ij}, \quad i = 0,\ldots,n,$$

where the reference bandwidth g is chosen such that $h, g \to 0$, $\frac{h}{g} \to 0$ for $n \to \infty$. Using the bootstrap data, we calculate kernel estimates in the bootstrap world

$$\widehat{\mathbf{m}}^*_I(\mathbf{x}, h) = \frac{1}{N}\sum_{i,j=0}^{n} K_h(\mathbf{x} - \mathbf{x}_{ij})\mathbf{Y}^*_{ij}, \qquad \widehat{\mathbf{m}}^*_{II}(\mathbf{x}, h) = \frac{1}{N}\sum_{i,j=0}^{n} K_h(\mathbf{x} - \mathbf{x}_{ij})\tilde{\mathbf{Y}}^*_{ij},$$

Mark that, e.g., $\widehat{m}^*_I(x,h)$ estimates $\widehat{m}_I(x,g)$ in the bootstrap world, and analogously for $\widehat{m}^*_{II}(x,h), \widehat{m}_{II}(x,g)$. Now, the bootstrap test statistic is constructed as follows.

$$T^*_N = Nh \int\int \left(\widehat{\mathbf{m}}^*_I(\mathbf{x}, h) - \widehat{\mathbf{m}}^*_{II}(\mathbf{x}, h)\right)^2 dx_1 dx_2$$

7 Appendix

As we shall use the following approximation result for sums by integrals repeatedly, we formulate it as a lemma.

Lemma 1 *For any Lipschitz function g on $[a,b]$ with Lipschitz constant L*

$$\left| \int_a^b g(x)dx - \frac{b-a}{n}\sum_{j=1}^{n} g(x_j) \right| \le \frac{L}{n}$$

with $x_0 = a, x_j = x_{j-1} + \frac{b-a}{n}, j = 1, 2, ..., n$.

Proof of Proposition 1:
The proof of this proposition proceeds along the same lines as in Härdle and Mammen [12]. Using

$$m_{II}(.) = m_I(.) + c_n\Delta_n(.)$$

and defining $\eta_i = \tilde{\varepsilon}_i - \varepsilon_i$ for $i = 0, ..., n$, we get

$$\begin{aligned} T_n &= \\ &= n\sqrt{h}\int (\widehat{m}_{II}(x,h) - \widehat{m}_I(x,h))^2 dx \\ &= n\sqrt{h}\int \Big[\frac{1}{n}\sum_{i=0}^{n} K_h(x_i - x)\{m_{II}(x_i) + \tilde{\varepsilon}_i - m_I(x_i) - \varepsilon_i\}\Big]^2 dx \\ &= \frac{\sqrt{h}}{n}\int \Big[U_{n,1}(x) + U_{n,2}(x)\Big]^2 dx \end{aligned}$$

with

$$U_{n,1}(x) = \sum_{i=0}^{n} K_h(x_i - x)c_n\Delta_n(x_i), \qquad U_{n,2}(x) = \sum_{i=0}^{n} K_h(x_i - x)\eta_i.$$

a.) First, we investigate the asymptotic behaviour of $U_{n,1}(x)$. By our smoothness assumptions on K, m_I, m_{II}, we have that K_h and $\Delta_n = c_n^{-1}(m_{II} - m_I)$ are Lipschitz continuous with Lipschitz constants of order h^{-2} and c_n^{-1} respectively. Therefore, $K_h(u - x)\Delta_n(u)$ is Lipschitz continuous in u with a constant of order $\max(h^{-2}, (hc_n)^{-1})$, as K, Δ_n are bounded. By Lemma 1 the approximation error of the sum by the integral is of order

$$\frac{1}{n}\max(h^{-2}, (hc_n)^{-1}) = \max\Big(\frac{1}{nh^2}, \frac{1}{n^{1/2}h^{3/4}}\Big)$$

and that is of order $n^{-7/20}$ by (K2). We get

$$\begin{aligned} \Big(\frac{\sqrt{h}}{n}\Big)^{1/2} U_{n,1}(x) &= \frac{1}{n}\sum_{i=0}^{n} K_h(x_i - x)\Delta_n(x_i) \\ &= \int K_h(u - x)\Delta_n(u)du + O(n^{-7/20}) \\ &= K_h * \Delta_n(x) + O(n^{-7/20}) \end{aligned}$$

We remark that $K_h * \Delta_n(x)$ is uniformly bounded by (A2) and (K1). Therefore, we also get

$$\frac{\sqrt{h}}{n}\int U_{n,1}^2(x)dx = \int (K_h * \Delta_n(x))^2 dx + O(n^{-7/20}) = B_h^1 + O(n^{-7/20}).$$

b.) As a next step, we investigate $U_{n,2}(x)$. First, we note that the η_i are independent with mean zero and

$$\mathrm{var}(\eta_i) = \mathrm{var}(\varepsilon_i) + \mathrm{var}(\tilde{\varepsilon}_i) = 2\sigma^2(x_i)$$

Then, we decompose

$$\begin{aligned} U_{n,2}^2(x) &= \Big(\sum_{i=0}^n K_h(x_i - x)\eta_i\Big)^2 \\ &= \sum_{i=0}^n K_h^2(x_i - x)\eta_i^2 + 2\sum_{i<j} K_h(x_i - x)K_h(x_j - x)\eta_i\eta_j \\ &= V_{n,2} + W_{n,2} \end{aligned}$$

$K_h^2(u-x)$ is Lipschitz continuous in u with Lipschitz constant of order h^{-3}, and $\sigma^2(u)$ is bounded and Lipschitz continuous by (A3). Therefore, using again Lemma 1

$$\begin{aligned} \mathrm{E}\ \frac{\sqrt{h}}{n} V_{n,2}(x) &= 2\frac{\sqrt{h}}{n}\sum_{i=0}^n K_h^2(x_i - x)\sigma^2(x_i) \\ &= 2\sqrt{h}\int_0^1 K_h^2(u-x)\sigma^2(u)du + O\Big(\frac{1}{nh^{5/2}}\Big) \\ &= \frac{2}{\sqrt{h}}\int K^2(y)\sigma^2(x+hy)dy + O\Big(\frac{1}{\sqrt{n}}\Big) \\ &= \frac{2}{\sqrt{h}}\sigma^2(x)\int K^2(y)dy + O\Big(\sqrt{h}\Big) \end{aligned}$$

where we have used (K2) and, for the last equation, that, by (A3), $|\sigma^2(x+hy) - \sigma^2(x)| = h|y|O(1) = O(h)$ for all $y \in [-1,1]$, i.e. in the support of K. Remark, that this result holds uniformly in x. We get

$$\begin{aligned} \mathrm{E}\ \frac{\sqrt{h}}{n}\int V_{n,2}(x)dx &= \frac{2}{\sqrt{h}}\int \sigma^2(x)dx\int K^2(u)du + O(\sqrt{h}) \\ &= B_h^0 + O(\sqrt{h}) \end{aligned}$$

Now, as $\mathrm{var}(\eta_i^2)$ is bounded uniformly in i by (A3) and our assumptions on the 4^{th} moments of $\varepsilon_i, \tilde{\varepsilon}_i$, we get, using (K1),

$$\begin{aligned} \mathrm{var}\Big(\frac{\sqrt{h}}{n}\int V_{n,2}(x)dx\Big) &= \frac{h}{n^2}\mathrm{var}\Big(\sum_{i=0}^n \int K_h^2(x_i - x)dx\ \eta_i^2\Big) \\ &= \frac{h}{n^2}\sum_{i=0}^n \Big(\int K_h^2(x_i - x)dx\Big)^2 \mathrm{var}(\eta_i^2) \\ &\le \frac{h}{n^2}\sum_{i=0}^n \Big(\frac{1}{h}\int K_h(x_i - x)dx\Big)^2 O(1) \\ &= O\Big(\frac{1}{nh}\Big) \end{aligned}$$

As, by (K2), $(nh)^{-1/2} \sim h^2$, we conclude

$$\frac{\sqrt{h}}{n}\int V_{n,2}(x)dx = B_h^0 + O(\sqrt{h}) + O_p(h^2)$$

c.) Now, we consider the term involving $W_{n,2}(x)$, and we prove

$$S_n = \frac{\sqrt{h}}{n} \int W_{n,2}(x)dx \to \mathcal{N}\Big(0, \sigma_T^2\Big) \tag{5}$$

First, put

$$L_{ijn} = \begin{cases} \sqrt{h} \int_0^1 K_h(x_i - x)K_h(x_j - x)dx & \text{if } i \neq j = 0, ..., n, \\ 0 & \text{if } i = j, \end{cases}$$

$$s_{ijn} = \frac{1}{n} L_{ijn} \eta_i \eta_j,$$

such that

$$S_n = \sum_{i,j} s_{ijn} = \frac{1}{n} \sum_{i \neq j} L_{ijn} \eta_i \eta_j.$$

According to Theorem 2.1 in de Jong [15] for (5) it suffices to prove

$$\text{var}(S_n) \to \sigma_T^2, \tag{6}$$

$$\frac{\max_{1 \leq i \leq n} \sum_{j=0}^n \text{var}(s_{ijn})}{\text{var}(S_n)} \to 0, \tag{7}$$

$$\frac{\text{E } S_n^4}{(\text{var}(S_n))^2} \to 3, \tag{8}$$

We have $\text{E}S_n = 0$, as $\eta_i \eta_j$ are independent and, for $i \neq j$, $\text{E}(\eta_i \eta_j) = \text{E}\eta_i \text{ E}\eta_j = 0$. Therefore,

$$\text{var}(S_n) = \text{E}S_n^2 = \frac{1}{n^2} \sum_{i \neq j} \sum_{l \neq k} L_{ijn} L_{kln} \text{E}(\eta_i \eta_j \eta_k \eta_l)$$

Now, we use the abbreviation

$$\ell_n(x, y) = \int_0^1 K_h(x - u)K_h(y - u)du = \begin{cases} 0, & \text{if } |x - y| > 2h, \\ O\big(h^{-1}\big), & \text{otherwise,} \end{cases} \tag{9}$$

such that $L_{ijn} = \sqrt{h} \ell_n(x_i, x_j)$ for $i \neq j$. We have used that K_h has support $[-h, h]$ and integrates to 1. As $\text{E}(\eta_i \eta_j \eta_k \eta_l) = 0$ in the double sum above except for $i = k, j = l$ or $i = l, j = k$ we get

$$\begin{aligned} \text{var}(S_n) &= \frac{2}{n^2} \sum_{i \neq j} L_{ijn}^2 \text{E } \eta_i^2 \eta_j^2 = \frac{8}{n^2} \sum_{i \neq j} L_{ijn}^2 \sigma^2(x_i) \sigma^2(x_j) \\ &= 8 \frac{h}{n^2} \sum_{i \neq j} \big(\ell_n(x_i, x_j)\big)^2 \sigma^2(x_i) \sigma^2(x_j) \\ &= 8h \int \int \big(\ell_n(x, y)\big)^2 \sigma^2(x) \sigma^2(y) dx dy + O\Big(\frac{1}{nh^2}\Big) \end{aligned} \tag{10}$$

where we have used Lemma 1 again and the fact that $\big(\ell_n(x, y)\big)^2$ is Lipschitz in x and y with a constant of order $O(h^{-3})$. Now, as K has support $[-1, 1]$,

$$\begin{aligned} \ell_n(x, y) &= \frac{1}{h} \int_{-x/h}^{(1-x)/h} K(z) K\big(z - \frac{x - y}{h}\big) dz \\ &= \frac{1}{h} K^{(*2)}\big(\frac{x - y}{h}\big) \qquad \text{for } h \leq x \leq 1 - h. \end{aligned}$$

As K is a probability density, the same holds for $K^{(*2)}$, and the latter satisfies (K1) too except that its support is $[-2,2]$. We get for $h \le x \le 1-h$

$$\int_0^1 \big(\ell_n(x,y)\big)^2 dy = \int_0^1 \Big(\frac{1}{h}K^{(*2)}\big(\frac{x-y}{h}\big)\Big)^2 dy = \frac{1}{h}\int \big(K^{(*2)}(z)\big)^2 dz = \frac{1}{h}K^{(*4)}(0),$$

and, using Lipschitz continuity of σ^2 with Lipschitz constant, say, L_σ and (9),

$$\begin{aligned}\Big|h\int_0^1 \sigma^2(y)\big(\ell_n(x,y)\big)^2 dy - \sigma^2(x)K^{(*4)}(0)\Big| &= \Big|h\int_0^1 \big(\sigma^2(y)-\sigma^2(x)\big)\big(\ell_n(x,y)\big)^2 dy\Big| \\ &\le 2h^2 L_\sigma \int_0^1 \big(\ell_n(x,y)\big)^2 dy \\ &= 2hL_\sigma K^{(*4)}(0) = O(h).\end{aligned}$$

Therefore, for $n \to \infty$,

$$\mathrm{var}(S_n) = 8K^{(*4)}(0)\int_h^{1-h} \sigma^4(x)dx + o(1) \to \sigma_T^2.$$

(6) follows. Remark that for $i \ne j$

$$\mathrm{var}(s_{ijn}) = \frac{1}{n^2}L_{ijn}^2 \mathrm{var}(\eta_i\eta_j) = \frac{1}{n^2}L_{ijn}^2 \ \mathrm{E}\ \eta_i^2 \ \mathrm{E}\ \eta_j^2 = \frac{4}{n^2}L_{ijn}^2\ \sigma^2(x_i)\sigma^2(x_j) = O\big(\frac{1}{n^2h}\big)$$

uniformly in i,j, using (9) and boundedness of $\sigma^2(x)$. (7) follows immediately as $nh \to \infty$.

Considering now (8), we have

$$\begin{aligned}\mathrm{E}\ S_n^4 &= \sum_{i,j,k,l,\mu,\nu,\kappa,\lambda} \mathrm{E}\ s_{ijn}s_{kln}s_{\mu\nu n}s_{\kappa\lambda n} \\ &= \frac{1}{n^4}\sum_{i,j,k,l,\mu,\nu,\kappa,\lambda} L_{ijn}L_{kln}L_{\mu\nu n}L_{\kappa\lambda n}\mathrm{E}\big(\eta_i\eta_j\eta_k\eta_l\eta_\mu\eta_\nu\eta_\kappa\eta_\lambda\big)\end{aligned}$$

The terms with $i=j, k=l$ etc. vanish by definition of L_{ijn}. Also, by independence of the η_i, the eight fold expectation vanishes if one index appears only once. So, typical terms which are not vanishing are of the form, using $s_{ijn} = s_{jin}$,

a) E $s_{ijn}^4, i \ne j$,

b) E $s_{ijn}^2 s_{ikn}^2, i \ne j \ne k$,

c) E $s_{ijn}^2 s_{kln}^2, i \ne j \ne k \ne l$,

d) E $s_{ijn}^2 s_{ikn}s_{jkn}, i \ne j \ne k$,

e) E $s_{ijn}s_{jkn}s_{kln}s_{lin}, i \ne j \ne k \ne l$

(compare also to the proof of Proposition 1 of Härdle and Mammen [12]). Let us first consider the number of terms like c) in detail. In the expansion of E S_n^4, they appear if $\{i,j\} = \{\mu,\nu\}, \{k,l\} = \{\kappa,\lambda\}$ or $\{i,j\} = \{\kappa,\lambda\}, \{k,l\} = \{\mu,\nu\}$ or $\{i,j\} = \{k,l\}, \{\mu,\nu\} = \{\kappa,\lambda\}$. For each of those 3 cases, we have 4 possibilities for choosing the indices, e.g. for the first case $(i,j) = (\mu,\nu), (k,l) = (\kappa,\lambda)$ or $(i,j) = (\nu,\mu), (k,l) = (\kappa,\lambda)$ or $(i,j) = (\mu,\nu), (k,l) = (\lambda,\kappa)$

or $(i,j)=(\nu,\mu),(k,l)=(\lambda,\kappa)$. So for each choice of $i\neq j\neq k\neq l$ we have $N_c=12$ terms of form c) in the expansion of $\mathrm{E}\ S_n^4$. Let N_a,N_b,N_d,N_e denote the corresponding number of terms of form a), b), d) and e); they could be derived in the same manner as for c), but this will turn out to be unnecessary for our argument. So, we have

$$\begin{aligned}\mathrm{E}\ S_n^4 &= N_a\sum_{i\neq j}\mathrm{E}\ s_{ijn}^4+N_b\sum_{i\neq j\neq k}\mathrm{E}\ s_{ijn}^2 s_{ikn}^2+12\sum_{i\neq j\neq k\neq l}\mathrm{E}\ s_{ijn}^2 s_{kln}^2\\ &\quad +N_d\sum_{i\neq j\neq k}\mathrm{E}\ s_{ijn}^2 s_{ikn}s_{jkn}+N_e\sum_{i\neq j\neq k\neq l}\mathrm{E}\ s_{ijn}s_{jkn}s_{kln}s_{lin}\\ &= N_aE_a+N_bE_b+12E_c+N_dE_d+N_eE_e\end{aligned}$$

We will show that all other terms are of smaller order than the third one. Using (9),

$$\begin{aligned}E_a&=\frac{1}{n^4}\sum_{i\neq j}L_{ijn}^4\mathrm{E}\ \eta_i^4\eta_j^4=\frac{1}{n^4}\sum_{i\neq j}L_{ijn}^4\mathrm{E}\ \eta_i^4\mathrm{E}\ \eta_j^4\\ &=O\Big(\frac{1}{h^2}\Big)\frac{1}{n^4}\sum_{i=0}^{n}\sum_{|j-i|\leq n2h}1=O\Big(\frac{1}{n^2h}\Big),\end{aligned}$$

$$\begin{aligned}E_b&=\frac{1}{n^4}\sum_{i\neq j\neq k}L_{ijn}^2L_{ikn}^2\mathrm{E}\ \eta_i^4\eta_j^2\eta_k^2=\frac{1}{n^4}\sum_{i\neq j\neq k}L_{ijn}^2L_{ikn}^2\mathrm{E}\ \eta_i^4\mathrm{E}\ \eta_j^2\mathrm{E}\ \eta_k^2\\ &=O\Big(\frac{1}{h^2}\Big)\frac{1}{n^4}\sum_{i=0}^{n}\sum_{|j-i|\leq n2h}\sum_{|k-i|\leq n2h}1=O\Big(\frac{1}{n}\Big),\end{aligned}$$

$$\begin{aligned}E_d&=\frac{1}{n^4}\sum_{i\neq j\neq k}L_{ijn}^2L_{ikn}L_{jkn}\mathrm{E}\ \eta_i^3\eta_j^3\eta_k^2=\frac{1}{n^4}\sum_{i\neq j\neq k}L_{ijn}^2L_{ikn}L_{jkn}\mathrm{E}\ \eta_i^3\mathrm{E}\ \eta_j^3\mathrm{E}\ \eta_k^2\\ &=O\Big(\frac{1}{h^2}\Big)\frac{1}{n^4}\sum_{i=0}^{n}\sum_{|j-i|\leq n2h}\sum_{|k-i|\leq n2h,|k-j|\leq n2h}1=O\Big(\frac{1}{n}\Big),\end{aligned}$$

$$\begin{aligned}E_e&=\frac{1}{n^4}\sum_{i\neq j\neq k\neq l}L_{ijn}L_{jkn}L_{kln}L_{lin}\mathrm{E}\ \eta_i^2\eta_j^2\eta_k^2\eta_l^2\\ &=O\Big(\frac{1}{h^2}\Big)\frac{1}{n^4}\sum_{i=0}^{n}\sum_{|j-i|\leq n2h}\sum_{|l-i|\leq n2h}\sum_{|k-j|\leq n2h,|k-l|\leq n2h}1=O\big(h\big).\end{aligned}$$

Finally, using similar arguments as in showing $E_a,E_b,E_d,E_e=o(1)$,

$$\begin{aligned}12E_c&=12\frac{1}{n^4}\sum_{i\neq j\neq k\neq l}L_{ijn}^2L_{kln}^2\mathrm{E}\ \eta_i^2\eta_j^2\eta_k^2\eta_l^2=12\frac{1}{n^4}\sum_{i\neq j}\sum_{k\neq l}L_{ijn}^2L_{kln}^2\mathrm{E}\ \eta_i^2\eta_j^2\mathrm{E}\ \eta_k^2\eta_l^2+o(1)\\ &=3\big(\mathrm{var}(S_n)\big)^2+o(1)\to 3\sigma_T^4\end{aligned}$$

by (10). Therefore, $\mathrm{E}\ S_n^4=3\big(\mathrm{var}(S_n)\big)^2+o(1)$, and (8) follows.

d) Recall the decomposition of T_n at the beginning of the proof, i.e.

$$T_n = \frac{\sqrt{h}}{n}\int U_{n,1}^2(x)dx + \frac{\sqrt{h}}{n}\int U_{n,2}^2(x)dx + \frac{2\sqrt{h}}{n}\int U_{n,1}(x)U_{n,2}(x)dx.$$

By a), b), c), the first two terms coincide with $B_h^1+B_h^0+S_n+o_p(1) \sim B_h+S_n$, and we get that T_n is asymptotically $\mathcal{N}(B_h, \sigma_T^2)$-distributed if we show that the third term is asymptotically negligible. Let

$$\begin{aligned}
S_n' &= \frac{\sqrt{h}}{n}\int U_{n,1}(x)U_{n,2}(x)dx \\
&= \left(\frac{\sqrt{h}}{n}\right)^{1/2}\frac{1}{n}\int\sum_{i,j} K_h(x_i-x)K_h(x_j-x)\Delta_n(x_i)\eta_j dx \\
&= \left(\frac{\sqrt{h}}{n}\right)^{1/2}\frac{1}{n}\sum_{i,j}\ell(x_i,x_j)\Delta_n(x_i)\eta_j
\end{aligned}$$

We have E $S_n'=0$ and, by independence of the η_j and using (9) again,

$$\begin{aligned}
\text{E }(S_n')^2 &= \frac{\sqrt{h}}{n^3}\sum_{i,j,k}\ell(x_i,x_j)\ell(x_k,x_j)\Delta_n(x_i)\Delta_n(x_k)\sigma^2(x_j) \\
&= \frac{\sqrt{h}}{n^3}\sum_{i=0}^{n}\sum_{|i-j|\le 2nh}\sum_{|k-j|\le 2nh} O\left(\frac{1}{h^2}\right) = O(\sqrt{h})
\end{aligned}$$

Therefore $S_n' \to 0$ in probability. ∎

Proof of Theorem 1:
The proof goes along the same line of arguments as in Proposition 1. In particular, (E1) implies $\sup_i \varepsilon_i^2, \sup_i \tilde{\varepsilon}_i^2 = O_p(\log n)$ and E ε_i^8, E $\tilde{\varepsilon}_i^8 \le const < \infty$ uniformly in $i = 1, 2, \ldots$. This can be used to prove the conditions of the theorem of de Jong [15] refered to above in part c) of the proof of Proposition 1. ∎

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