Resolving Parameter Dependences for Interval Analysis of Linear Analog Circuits

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***A*bstract – Reliable methods for the analysis of tolerance-affected analog circuits are of great importance in nowadays microelectronics. Because of variations during the production, parameters of real circuits differ from the values specified in the design process. If upper and lower bounds of such parameter variations are known, interval analysis can be applied to analyze the effect of these deviations, but yet this was restricted to circuit equations without interdependences. Hence, for an efficient application of interval methods, it is crucial to regard possible dependences in circuit equations. Part and parcel of of this strategy is the handling of symbolic fill-in patterns of uncertain components. A variant of the *Sherman-Morrison formula* can be utilized for efficient frequency-response analysis of tolerance-affected linear analog circuits.**

1 MOTIVATION

Numerical simulations of analog circuits can be used to analyze a circuit's behaviour without the need for a physical implementation. But actual circuit properties may differ from the results obtained by floating-point simulations, due to errors caused by rounding, component tolerances, and simplified models. Simulations based on interval arithmetic can be used as a unified framework to bound all these errors, but tend to be too conservative. In this paper a new approach for computing tight bounds to frequency response of tolerance-affected analog circuits is described.

The behaviour of analog circuits can be described by a system of parameter-dependent linear or nonlinear equations. A symbolic setup of the equation system allows for assigning unique symbols to each circuit component parameter. The resulting circuit equations cannot be analyzed in a pure symbolic way: In the nonlinear case the system might not be solvable in a symbolic manner, but yet in the linear case the result may be of large complexity. In order to analyze the behavior of such an analog circuit using a simulator or a numerical solver, the symbols representing netlist elements have to be replaced by the corresponding numerical values, according to a given design point.

The numerical approach has two major drawbacks: First of all, for an efficient numerical treatment of the equation system, all numerical values have to be converted into floating-point numbers. This may lead to growing of the overall error, due to rounding of the numerical values in each solving step. Another problem is caused by the fact that a design point, i. e. a fixed set of parameters, may be defined in advance, but one cannot ensure a priori that the desired properties will exactly be met during manufacturing of the actual circuit. Component tolerances will always lead to variations of a circuit's properties, which may result in effects not expected from the results of the numerical simulation. While rounding errors could be reduced or even completely avoided by a sophisticated treatment of the equation system, the latter problem cannot be overcome within a single numerical simulation. Using a statistical method, like the Monte Carlo approach, the parameter variations of the production process may be simulated [1]. But this results in a large number of simulations and does not yield guarateed solutions. Simulation based on *interval arithmetic* can be used as a unified framework for both problems.

This paper starts with a brief introduction to the principles of interval computations and the relevance of interdependences to the accuracy of interval-valued results. Then mathematical methods especially tuned to handle systems arising from frequency-response analysis of linear analog circuits are described. This includes an approach, which is capable of computing tight bounds to variations due to a large number of tolerance-affected components. Finally, the results are illustrated by giving example applications.

2 INTERVAL ARITHMETIC

2.1 Basics

If upper and lower bounds for the uncertain parameters can be determined, these can be interpreted as the endpoints $\underline{x}$, $\overline{x}$ of a closed interval $[\underline{x},\overline{x}] \subseteq \mathbb{R}$. A vector of intervals (*box*) is denoted as $[\mathbf{x}]$. The principles of interval arithmetic (e.g. [2, 3]) utilize the fact, that during evaluation any expression is constructed by subsequent calls of elementary binary operations $(+,-,*,/)$ and basic functions like *sin*, *cos*, *log*, e^x and x^n, where the *intervalization* of binary operators is

$$\begin{aligned}
&[\underline{x},\overline{x}] \diamond [\underline{y},\overline{y}] = [\underline{z},\overline{z}],\ \text{for} \diamond \in \{+,-,*,/\},\\
&\text{with } \underline{z} = \min\{\underline{x}\diamond\underline{y}, \underline{x}\diamond\overline{y}, \overline{x}\diamond\underline{y}, \overline{x}\diamond\overline{y}\},\\
&\text{and } \overline{z} = \max\{\underline{x}\diamond\underline{y}, \underline{x}\diamond\overline{y}, \overline{x}\diamond\underline{y}, \overline{x}\diamond\overline{y}\},
\end{aligned} \tag{1}$$

if the operation is valid for all values in range. Functions like $e^{[\underline{x},\overline{x}]}$ and $[\underline{x},\overline{x}]^n$ can be defined in an ana-

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logous manner. The intervalization of any monotonic or piecewise monotonic elementary function is computed by evaluating the function on a finite set of *special points*, consisting of the interval's endpoints and local extrema.

For bounding the range of a more complex expression we have to assign a corresponding *interval extension* to it. An interval-valued function $[f]$ is called an interval extension of the real-valued function f, if $[f]([\underline{x}_1,\overline{x}_1],\ldots,[\underline{x}_n,\overline{x}_n]) \supseteq \{f(y_1,\ldots,y_n) \,|\, y_i \in [\underline{x}_i,\overline{x}_i]\}$. The interval extension obtained by replacing real operations and elementary functions by their interval-valued equivalents is called *natural interval extension* [2].

2.2 The Dependence Problem and Symbolic Solutions

The major drawback in using interval computations is caused by the *dependence problem*. While computations using independent parameters will return tight bounds to the exact range of the function, two or more occurrences of the same parameter during the evaluation phase will result in too conservative estimations. For instance, consider the voltage divider circuit in Figure 1.

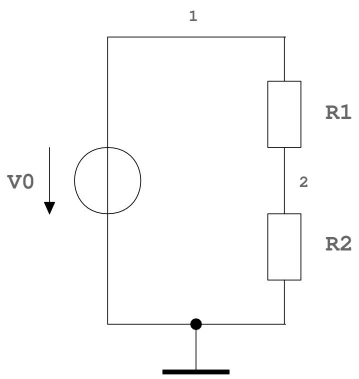

Figure 1: Voltage divider circuit.

Its behaviour can be described by the matrix equation

$$\begin{pmatrix} \frac{1}{R_1} & -\frac{1}{R_1} & 1 \\ -\frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_2} & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} V_1 \\ V_2 \\ I_{V_0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ V_0 \end{pmatrix}. \qquad (2)$$

In this case, one can explicitely find a symbolic solution i. e. for each variable we obtain an expression in terms of V_0, R_1, and R_2

$$V_1 = V_0\,,\; V_2 = \frac{R_2 \cdot V_0}{R_1 + R_2}\,,\text{ and } I_{V_0} = -\frac{V_0}{R_1 + R_2}\,.$$

Assuming $V_0 = 1\,\mathrm{V}$ and uncertain parameter values $R_1/1\,\Omega \in [9,11]$, and $R_2/1\,\Omega \in [90,110]$, interval arithmetic can be used to compute rough bounds to the range with respect to these settings, that is $V_1 = 1\,\mathrm{V}$ and $V_2/1\,\mathrm{V} \in [0.743, 1.112]$ as well as $I_{V_0}/1\,\mathrm{mA} \in [-10.1, -8.27]$. But here the voltage V_2 can be bounded more accurately. Using the equivalent reformulation $V_2 = \frac{V_0}{1+R_1/R_2}$ to compute the interval results, the tighter range $V_2/1\,\mathrm{V} \in [0.891, 0.925]$ is obtained. Since each uncertain parameter occurs only once, this is the best possible result. Therefore, simply replacing numerical solvers by an interval version would soon lead to rather useless results, consisting of large parts of the original search region.

Several algorithms have been developed to solve linear and nonlinear interval-valued equation systems [2, 3]. These behave well if we are dealing with intervals of small width and little dependence of the interval-valued terms of the expressions involved, but real-life applications will require the treatment of wider intervals and parameters of multiple occurrences. For our purpose interval algorithms have to be tuned for solving equation systems resulting from industrial analog circuits.

3 MATHEMATICAL METHODS

In this section we restrict ourselves to the treatment of linear circuit elements.

Earlier efforts to solve interval-valued linear circuit equations were restricted to formulations, in which each matrix element varies independently [4]. Therefore a new approach was developed to cope with multiple occurrences of parameters: we have already seen in [5] that a resistive circuit may be represented by the parametric matrix equation $\mathbf{A}(\mathbf{p}) \cdot \mathbf{x} = \mathbf{b}$, whose parameter dependence can be written as a sequence of rank-one updates of a parameter-independent matrix $\mathbf{A}_0 \in \mathbb{R}^{n\times n}$:

$$\mathbf{A}(\mathbf{p}) = \mathbf{A}_0 + \sum_{i=1}^{n_\mathbf{p}} p_i \cdot (\mathbf{u}_i \cdot \mathbf{v}_i{}^\mathsf{T})\,, \qquad (3)$$

where $\mathbf{u}_i, \mathbf{v}_i \in \mathbb{R}^n$, and $\mathbf{A}(\mathbf{p})$ is invertible for all $\mathbf{p} \in [\mathbf{p}]$. In case that capacitances and inductivities are envolved, the structure is likewise, but the corresponding p_i may vary on the imaginary axis instead. This is not a restriction at all, because this structure is already inherent to a linear circuit: Using the *sparse tableau formulation* (*STA*) [6] to generate the linear circuit equations, each p_i will occur only once and $\mathbf{u}_i, \mathbf{v}_i$ are unit vectors, which define the corresponding matrix element. In the case of *modified nodal analysis* (*MNA*) [6] the matrices $p_i \cdot (\mathbf{u}_i \cdot \mathbf{v}_i{}^\mathsf{T})$ correspond to the well-known fill-in patterns used during equation setup. This form can be utilized for efficient solving of interval-valued circuit equations [7].

3.1 Methods for Resistive Circuits

The *Sherman-Morrison formula* [8] allows for inverting a perturbed matrix for a change to a given matrix without inverting the whole matrix again.

Theorem 3.1 (Sherman-Morrison) *Let* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *be invertible and* $\mathbf{u},\mathbf{v} \in \mathbb{R}^n$*. Then the matrix* $\mathbf{A}+\mathbf{u}\cdot\mathbf{v}^\mathsf{T}$ *is invertible if and only if* $1+\mathbf{v}^\mathsf{T}\mathbf{A}^{-1}\mathbf{u} \neq 0$*. In this case we have:*

$$\left(\mathbf{A}+\mathbf{u}\cdot\mathbf{v}^\mathsf{T}\right)^{-1} = \mathbf{A}^{-1} - \frac{1}{1+\mathbf{v}^\mathsf{T}\mathbf{A}^{-1}\mathbf{u}}\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^\mathsf{T}\mathbf{A}^{-1}. \quad (4)$$

Equation 4 has already been used in the field of analog circuit analysis for calculating the influence of a single matrix entry to the solution [9, 10]. It can immediately be used to analyze the pertubations of $\mathbf{A}(p)^{-1}\mathbf{b}$ with respect to to one uncertain parameter p, which may vary in the interval $[\underline{p},\overline{p}]$. For a desired design point $p_0 \in [\underline{p},\overline{p}]$ the corresponding fill-in pattern can then be written as $\mathbf{A}(p) = \mathbf{A}(p_0) + (p-p_0)\cdot\mathbf{u}\mathbf{v}^\mathsf{T}$. If the matrix $\mathbf{A}(p)$ is invertible for all p in the range, the conditions for the Sherman-Morrison formula are met. Hence, we have that $\mathbf{A}(p)^{-1}\mathbf{b}$ can be computed with respect to the design-point solution $\mathbf{x}_0 = \mathbf{A}(p_0)^{-1}\mathbf{b}$ as

$$\mathbf{x}_0 - \frac{p-p_0}{1+(p-p_0)\cdot\mathbf{v}^\mathsf{T}\mathbf{A}(p_0)^{-1}\mathbf{u}}\mathbf{A}(p_0)^{-1}\mathbf{u}\mathbf{v}^\mathsf{T}\mathbf{x}_0$$

for all $p \in [\underline{p},\overline{p}]$. Because of monotonic properties of the right-hand side the result must lie on the line between $\mathbf{A}(\underline{p})^{-1}\mathbf{b}$ and $\mathbf{A}(\overline{p})^{-1}\mathbf{b}$. Analogously, for several parameters $p_i \in [\underline{p}_i,\overline{p}_i]$, only the corner points

$$P := \left\{ \left(p_1,\ldots,p_{n_\mathbf{p}}\right) \middle| \, p_i \in \{\underline{p}_i,\overline{p}_i\} \right\} \quad (5)$$

have to be considered to obtain close bounds.

Note, that this is possible, only if regularity of $\mathbf{A}(\mathbf{p})$ can be established for all $\mathbf{p} \in [\mathbf{p}]$. A sufficient condition for this case is the constancy of $\operatorname{sign}\det\mathbf{A}(\mathbf{p})$ for all $\mathbf{p} \in P$.

Hence, the interval-valued problem can be solved by processing of those $2^{n_\mathbf{p}}$ linear systems corresponding to the corner points. The advantage of this approach is, that it does not need interval computations. Of course, interval arithmetic can be used to bound rounding errors. In the case that these do not have to be tracked, already existing numerical solvers, like those of analog circuit simulators [11, 12], can be utilized for tolerance analysis.

The approach described above is suitable for small parameter numbers $n_\mathbf{p}$ only, because the interval-valued problem is put down to the solution of $2^{n_\mathbf{p}}$ real-valued linear systems. In order to treat a large numbers of parameters, we use a kind of intervalization of the Sherman-Morrison formula to obtain a less accurate, but faster algorithm. As seen in the case of a single uncertain parameter $p \in [\underline{p},\overline{p}]$, the solution $\mathbf{A}(p)^{-1}\mathbf{b}$ equals

$$\mathbf{A}(p_0)^{-1}\mathbf{b} - \frac{p-p_0}{d(p)}\mathbf{A}(p_0)^{-1}\mathbf{u}\mathbf{v}^\mathsf{T}\mathbf{A}(p_0)^{-1}\mathbf{b}, \quad (6)$$

if $d(p) = 1+(p-p_0)\cdot\mathbf{v}^\mathsf{T}\mathbf{A}(p_0)^{-1}\mathbf{u}$ holds for fixed value $p_0 \in [\underline{p},\overline{p}]$ under the conditions of the Sherman-Morrison theorem. But the latter can also be applied, if we are able to show that $d(p) \neq 0$ for all $p \in [\underline{p},\overline{p}]$. This can efficiently be calculated by evaluating

$$d([\underline{p},\overline{p}]) = 1+([\underline{p},\overline{p}]-p_0)\cdot(\mathbf{v}^\mathsf{T}\mathbf{A}(p_0)^{-1}\mathbf{u}) \quad (7)$$

using interval arithmetic. Hence, it can be verified, whether $d([\underline{p},\overline{p}]) \not\ni 0$. If that is the case, the range of the factor $[\underline{m},\overline{m}] := (p-p_0)/d(p)$ over all $p \in [\underline{p},\overline{p}]$ is exactly bounded by

$$\begin{aligned} \underline{m} &= ((\underline{p}-p_0)^{-1} + \mathbf{v}^\mathsf{T}\mathbf{A}_0^{-1}\mathbf{u})^{-1} \\ \overline{m} &= ((\overline{p}-p_0)^{-1} + \mathbf{v}^\mathsf{T}\mathbf{A}_0^{-1}\mathbf{u})^{-1}. \end{aligned} \quad (8)$$

One can easily extend this approach to the case of several parameters, by successive application of the formulation for each fill-in pattern pattern $\mathbf{u}_i\cdot\mathbf{v}_i^\mathsf{T}$ corresponding to a parameter p_i. Since also intermediate values like $\mathbf{A}(\mathbf{p})^{-1}\mathbf{u}_i$ have to be updated accordingly, it results in a procedure of order $\mathscr{O}(n_\mathbf{p}^2)$. Hence, it is suited to problems with a lot of parameters, which is an advantage over the corner point method, but at the cost of reduced accuracy, because interdependences are not removed completely.

3.2 Methods for RLC Networks

The small-signal analysis of analog circuits can be achieved by solving complex-valued linear systems. The corresponding matrix system also emerges from superposition of fill-in patterns, which can be defined via real vectors, but instead of real-valued parameters like resistances or conductances, the parameters are typically of the form $C\cdot s$ (capacitance) or $\frac{1}{L\cdot s}$ (inductance), for a complex-valued Laplace frequency $s = 2\pi i f$, and uncertain values C and L, respectively.

For a fixed frequency f this results in a linear complex-valued system $\mathbf{C}\cdot\mathbf{y} = \mathbf{d}$, with $\mathbf{C} \in \mathbb{C}^{n\times n}$, right-hand side $\mathbf{d} \in \mathbb{C}^n$, and variables $\mathbf{y} \in \mathbb{C}^n$. In order to apply (real) interval techniques it is necessary to reformulate the complex-valued equation system by the following equivalent real representation:

$$\begin{gathered} \mathbf{A}\cdot\mathbf{x} = \mathbf{b}, \text{ with } \mathbf{A} = \begin{pmatrix} \operatorname{Re}\mathbf{C} & -\operatorname{Im}\mathbf{C} \\ \operatorname{Im}\mathbf{C} & \operatorname{Re}\mathbf{C} \end{pmatrix}, \\ \mathbf{x} = \begin{pmatrix} \operatorname{Re}\mathbf{y} \\ \operatorname{Im}\mathbf{y} \end{pmatrix}, \text{ and } \mathbf{b} = \begin{pmatrix} \operatorname{Re}\mathbf{d} \\ \operatorname{Im}\mathbf{d} \end{pmatrix}. \end{gathered} \quad (9)$$

In the real representation, matrix elements corresponding to the same parameter are spread over lower and upper parts of the system. Hence, the structure cannot be captured with a single fill-in pattern form only, but two of them will do in the following way

$$\mathbf{A}(\mathbf{p}) = \mathbf{A}_0 + \sum_{\nu=1}^{n_\mathbf{p}} p_\nu \cdot (\mathbf{u}_{\mathrm{up},\nu} \cdot \mathbf{v}_{\mathrm{up},\nu}^\mathsf{T} + \mathbf{u}_{\mathrm{low},\nu} \cdot \mathbf{v}_{\mathrm{low},\nu}^\mathsf{T}) . \tag{10}$$

Now, the methods developed for resistive systems can be applied, if all occuring fill-in patterns are treated independently. For this purpose, new parameters $p_{\mathrm{up},\nu}, p_{\mathrm{low},\nu}$ are introduced, such that the right-hand side of Equation 10 becomes

$$\mathbf{A}_0 + \sum_{\nu=1}^{n_\mathbf{p}} (p_{\mathrm{up},\nu} \cdot \mathbf{u}_{\mathrm{up},\nu} \cdot \mathbf{v}_{\mathrm{up},\nu}^\mathsf{T} + p_{\mathrm{low},\nu} \cdot \mathbf{u}_{\mathrm{low},\nu} \cdot \mathbf{v}_{\mathrm{low},\nu}^\mathsf{T}) .$$

Since, the dependence between lower and upper part of $\mathbf{A}(\mathbf{p})$ is lost, this approach leads to an overestimation of the range as illustrated in Figure 2.

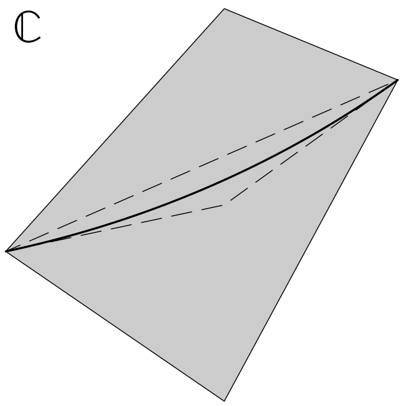

Figure 2: Variance of the solution with respect to one component parameter. Overestimation due to real representation (quadrangle) and desirable tight wrapping (dashed triangle).

The structure of corresponding fill-in patterns can be utilized to obtain the tighter wrapping of the solution set, shown by the dashed triangle in Figure 2. The pair of vectors $\mathbf{u}_{\mathrm{low},\nu}, \mathbf{v}_{\mathrm{low},\nu}$ on the one hand, and $\mathbf{u}_{\mathrm{up},\nu}, \mathbf{v}_{\mathrm{up},\nu}$ on the other, share the valuable property, that

$$\mathbf{u}_{\mathrm{low},\nu} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \mathbf{u}_{\mathrm{up},\nu} \tag{11}$$

and likewise for $\mathbf{v}_{\mathrm{low},\nu}$ and $\mathbf{v}_{\mathrm{up},\nu}$.

If the condition of Equation 11 holds for two pairs of vectors $\mathbf{u}_{\mathrm{low}}, \mathbf{u}_{\mathrm{up}}$, and ,$\mathbf{v}_{\mathrm{low}}, \mathbf{v}_{\mathrm{up}} \in \mathbb{R}^{2n}$, which represents the fill-in patterns of a single parameter p, and

$$\mathbf{A}(p_1, p_2) = \mathbf{A}_0 + p_1 \cdot \mathbf{u}_{\mathrm{low}} \cdot \mathbf{v}_{\mathrm{low}}^\mathsf{T} + p_2 \cdot \mathbf{u}_{\mathrm{up}} \cdot \mathbf{v}_{\mathrm{up}}^\mathsf{T} \tag{12}$$

for $\mathbf{A}_0 \in \mathbb{R}^{2n\times 2n}$ a real representation of a matrix in $\mathbb{C}^{n\times n}$, then the variation of $\mathbf{A}(p,p)^{-1} \cdot \mathbf{b}$ for values $p \in [\underline{p}, \overline{p}]$ can be bounded by a convex set defined by three points (triangle in Figure 2), as follows

$$\begin{array}{rcl} & & \mathbf{A}(p,p)^{-1} \cdot \mathbf{b} \in \mathrm{conv}\left(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_{\frac{1}{2}}\right) \\ \mathbf{x}_1 & = & \mathbf{A}(\underline{p}, \underline{p})^{-1} \cdot \mathbf{b} \\ \mathbf{x}_2 & = & \mathbf{A}(\overline{p}, \overline{p})^{-1} \cdot \mathbf{b} \\ \mathbf{x}_{\frac{1}{2}} & = & \frac{1}{2} \cdot \left(\mathbf{A}(\underline{p}, \overline{p})^{-1} \cdot \mathbf{b} + \mathbf{A}(\overline{p}, \underline{p})^{-1} \cdot \mathbf{b}\right) . \end{array} \tag{13}$$

Although Equation 13 can easily be generalized to the case of several parameters, it is not the method of choice for practical problems, because it would lead to $2^{2n_\mathbf{p}}$ linear systems to be solved. But in order to avoid such conservative bounds, as obtained by simply using the real representation, a combination with the ideas of resistive case is suitable.

For bounding all $\mathbf{A}(p,p)^{-1}\mathbf{b}$ for $p \in [\underline{p}, \overline{p}]$ the points $\mathbf{A}(\underline{p}, \underline{p})^{-1}\mathbf{b}$, $\mathbf{A}(\overline{p}, \overline{p})^{-1}\mathbf{b}$, $\mathbf{A}(\underline{p}, \overline{p})^{-1}\mathbf{b}$, and $\mathbf{A}(\overline{p}, \underline{p})^{-1}\mathbf{b}$ are needed by Equation 13. Starting at $(\underline{p}, \underline{p})$, an initial solution $\mathbf{A}(\underline{p}, \underline{p})^{-1}\mathbf{b}$ can be computed by plain linear system solving. In this case the condition

$$1 + (\overline{p} - \underline{p}) \cdot \mathbf{u}_{\mathrm{up}}^\mathsf{T} \mathbf{A}(\underline{p}, \underline{p})^{-1} \mathbf{u}_{\mathrm{up}} > 0 \tag{14}$$

implies that $\mathbf{A}(\underline{p}, p)$ is invertible for all p. Hence, the next point $\mathbf{A}(\underline{p}, \overline{p})^{-1}\mathbf{b}$ can be calculated using the Sherman-Morrison formula of Equation 4. Analogously, the vector $\mathbf{A}(\overline{p}, \underline{p})$ is obtained, and finally $\mathbf{A}(\overline{p}, \overline{p})$ may be generated by a second application of Sherman-Morrison, like in the case of a real-valued system with two parameters.

More than one parameter can be treated, if the sets of real-valued vectors are replaced by the smallest interval vector containing all points each time the treatment of the next parameter is finished. Therefore, from the second step on, all computations are done using interval arithmetic. Like in the resistive case several auxiliary results have to be updated accordingly.

Again the loss of accuracy due to wrapping intermediate results by interval vectors pays off by reducing the computational complexity order to $n_\mathbf{p}^2$.

4 EXAMPLE APPLICATIONS

The approach is demonstrated by analyzing the AC equations for the serial circuit of Figure 3. It has the same topological structure as the simple voltage divider of Figure 1, but one of the resistors is replaced by a capacitor.

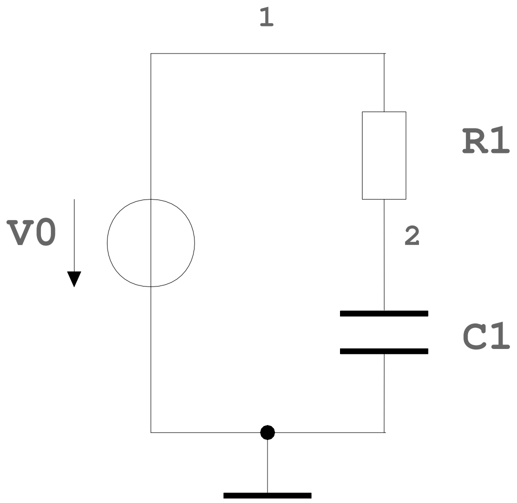

Figure 3: Serial circuit.

The same is true for the system of equations,

$$\begin{pmatrix} \frac{1}{R_1} & -\frac{1}{R_1} & 1 \\ -\frac{1}{R_1} & \frac{1}{R_1}+C_1 s & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} V_1 \\ V_2 \\ I_{V_0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ V_0 \end{pmatrix}, \quad (15)$$

which is to be treated for a constant frequency of 1 kHz, such that $s = 2\pi i \cdot 1000\,\mathrm{s}^{-1}$, with an exact independent voltage source of $V_0 = 1\,\mathrm{V}$, and two tolerance-affected parameters given as $C_1/1\,\mu\mathrm{F} \in [0.8, 1.2]$ as well as $R_1/1\,\Omega \in [80, 120]$.

The uncertain parameters in the sense of Equation 13 are defined as $p_1 = 1\,\Omega/R_1 \in [0.0083, 0.0125]$ and $p_2 = 2\pi \cdot 1000 \cdot C_1/1\,\mathrm{F} \in [0.0050, 0.0076]$. Application of the procedure leads to bounds for real and imaginary part of the current

$$\begin{array}{lcl} \mathrm{Re}\, I_{V_0}/1\,\mathrm{mA} & \in & [-4.26, -1.74] \\ \mathrm{Im}\, I_{V_0}/1\,\mathrm{mA} & \in & [-5.55, -3.69]\,. \end{array} \quad (16)$$

Absolute value and phase can be estimated from the corners of this box as $\left|I_{V_0}\right|/1\,\mathrm{mA} \in [4.3, 6.5]$, and $\varphi/1^\circ \in [-132.138, -111.906]$. This is an approximation, which gives sufficiently accurate bounds for practical applications (see also Section 3.3.2 of [4]).

The response for the frequencies from 1 Hz up to 1 MHz is compared to the actual results, which were approximated using a sufficiently large number of parameter sweeps. The curves denoting lower and upper bounds for both computations, respectively, indicate similar qualitative behavior. Although the results obtained with the interval method are less accurate, they allow to draw conclusions about the performance of the circuit in less computation time.

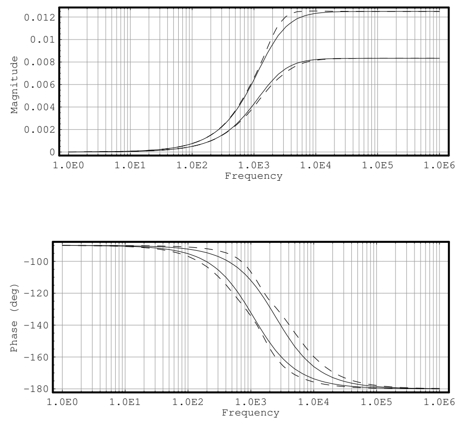

Figure 4: Interval Bode diagram of current I_{V_0}. Bounds (solid), and outer approximation using the interval method (dashed).

Following, we demonstrate the ability of the methodology to treat real-world examples. We consider the operational amplifier, whose schematics is illustrated in Figure 5. The small-signal behavior of the transistors is modeled by the simplified equivalent circuit of Figure 6, which consists of resistors, capacitors and controlled sources.

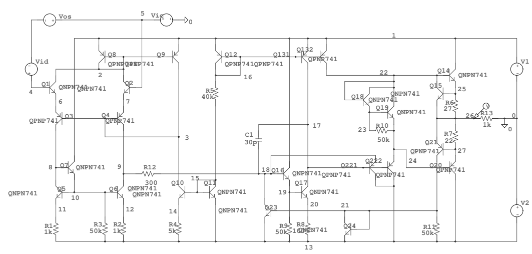

Figure 5: Operational amplifier $\mu A741$.

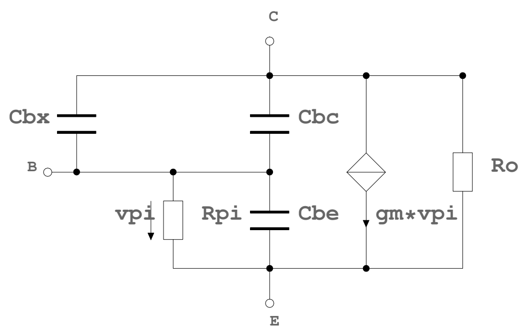

Figure 6: Simplified equivalent circuit diagram for a bipolar junction transistor.

Since the small-signal behaviour is only valid for a constant operating point, we will only vary such circuit elements, which do not change the operating point.

Hence, we assign tolerances of 20% to the capacitance values.

Then an interval-valued AC analysis is performed for 70 frequencies between 0.1 Hz and 1 MHz in about eleven minutes and 47 seconds. The frequency response of the output voltage at node 26 is presented in Figure 7. We obtain interpretable results: upper und lower bounds are very close to the curve, which corresponds to the original design point.

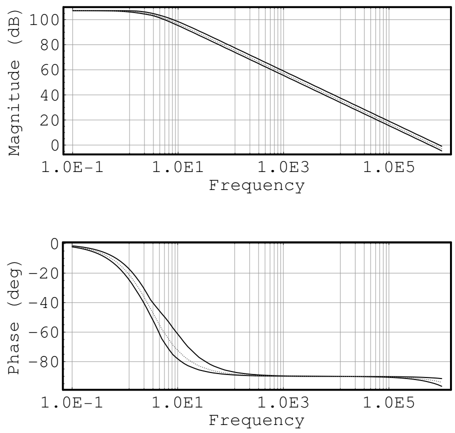

Figure 7: Bode diagram of the small-signal behavior of the operational amplifier. Lower and upper bounds (solid), frequency response with respect to original design point (dotted).

5 CONCLUSIONS

The methods described above were implemented as an extension to the toolbox *Analog Insydes* [12, 13], an add-on package to the computer algebra system *Mathematica* [14] for modeling, analysis, and design of analog circuits.

The techniques can be used to obtain meaningful bounds to the simulation results of analog circuits with uncertain parameters. As opposed to earlier attempts to use interval arithmetic in this area, which were restricted to the sparse tableau formulation, the dependence between parameters with multiple occurrences is treated accordingly.

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References

[1] V. Litkovski and M. Zwolinski. *VLSI Circuit Simulation and Optimization*. Chapman & Hall, London, UK, 1997.

[2] E. R. Hansen. *Global optimization using interval methods*, volume 165 of *Monographs and textbooks in pure and applied mathematics*. Marcel Dekker, New York, 1992.

[3] R. B. Kearfott. *Rigorous Global Search: Continuous Problems*, volume 13 of *Nonconvex optimization and its applications*. Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.

[4] L. V. Kolev. *Interval Methods for Circuit Analysis*. World Scientific, Singapore, 1993.

[5] Alexander Dreyer. Combination of symbolic and interval-numeric methods for analysis of analog circuits. In *Proc. 8th International Workshop on Symbolic Methods and Applications in Circuit Design (SMACD 2004)*, Wroclaw, Poland, Sep. 2004.

[6] J. Vlach and K. Singhal. *Computer Methods for Circuit Analysis and Design*. Van Nostrand Reinhold, New York, 2nd edition, 1993.

[7] Alexander Dreyer. *Interval Analysis of Analog Circuits with Component Tolerances*. PhD thesis, Kaiserslautern University of Technology, Kaiserslautern, Germany, 2005. Shaker Verlag, Aachen, germany.

[8] Jochen Werner. *Numerische Mathematik 1*. vieweg, Braunschweig, Germany, 1991.

[9] Eckhard Hennig. *Symbolic Approximation and Modeling Techniques for Analysis and Design of Analog Circuits*. Shaker Verlag, Aachen, Germany, 2000.

[10] Tim Wichmann. *Symbolische Reduktionsverfahren für nichtlineare DAE-Systeme*. Berichte aus der Mathematik. Shaker Verlag, Aachen, Germany, 2004.

[11] MicroSim Corporation. *MicroSim PSpice & Basics User's Guide*, 1996.

[12] Fraunhofer ITWM. Analog Insydes website. `http://www.analog-insydes.de`.

[13] Jochen Broz, Alexander Dreyer, Thomas Halfmann, Eckhard Hennig, Manfred Thole, and Tim Wichmann. *Analog Insydes - Release 2.1 Manual*. Fraunhofer-Institut für Techno- und Wirtschaftsmathematik, Kaiserslautern, Germany.

[14] S. Wolfram. *The Mathematica Book*, volume 4. Wolfram Media/Cambridge University Press, 4th edition, 1999.