PolyBoRi - A Framework for Gröbner Basis Computations with Boolean Polynomials

Michael Brickenstein

Mathematisches Forschungsinstitut Oberwolfach

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Fraunhofer Institute for Industrial Mathematics (ITWM)

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Strobl, Austria, June 26th, 2007

*Presentation

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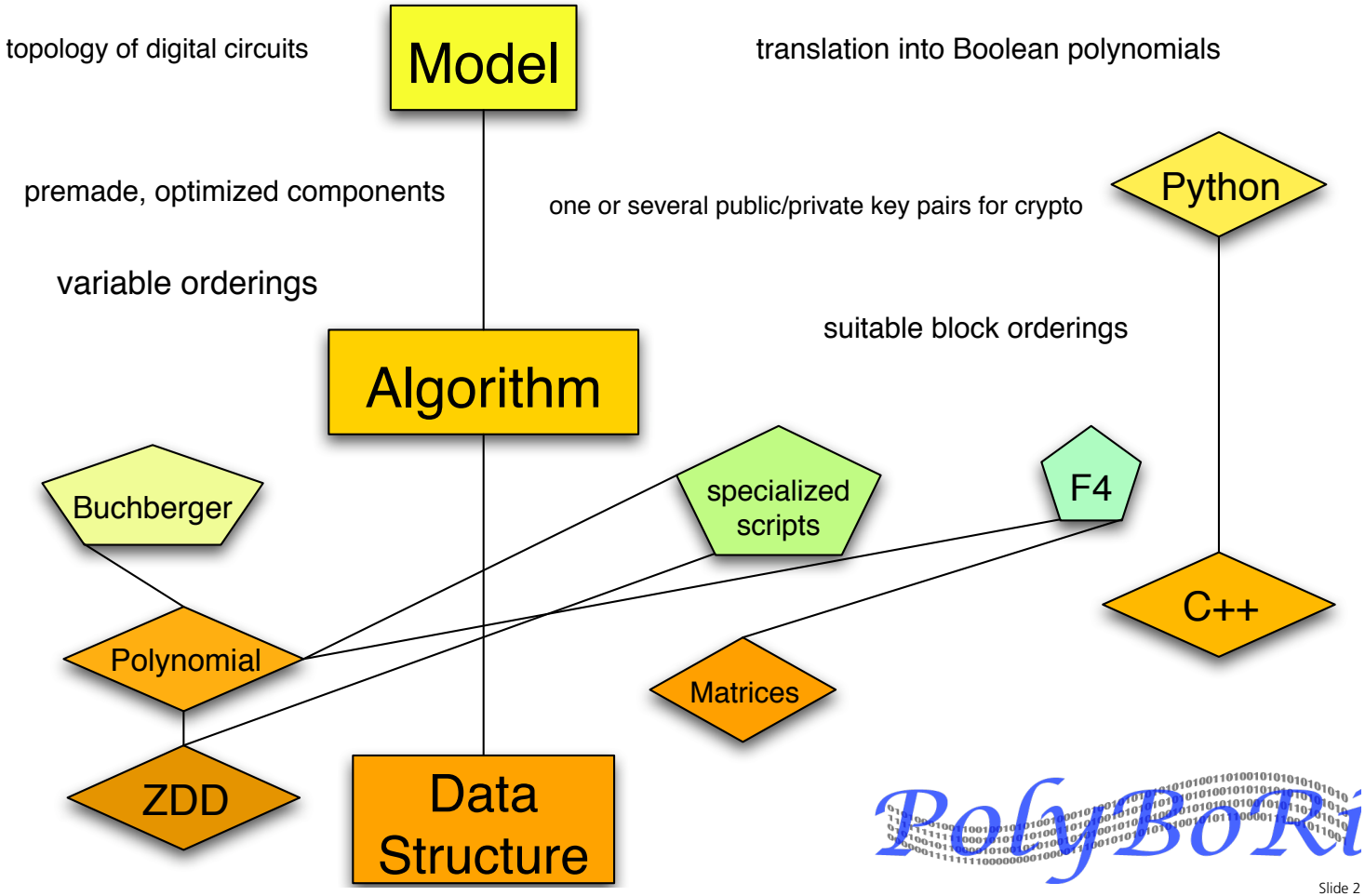
Basic Data

PolyBoRi	***Poly***nomials over ***Bo***olean ***Ri***ngs
DFG Project	"Development, implementation and application of mathematical-algebraic algorithms for formal verification of digital systems with arithmetic blocks"
Fraunhofer ITWM	Alexander Dreyer (Department Adaptive Systems)
University of Kaiserslautern	Algebra, Geometry and Computer Algebra Group (Prof. Greuel, Department of Mathematics)
Doctoral thesis	Michael Brickenstein (Mathematisches Forschungsinstitut Oberwolfach)

Overview

Optimization on many levels



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Digital systems verification and Boolean Polynomials

Interprete Boolean expressions as polynomials over $\mathbb{Z}_2$

logical operations $\longrightarrow$ arithmetical operations and $\{\mathsf{true}, \mathsf{false}\} \longrightarrow \{0, 1\}$

$$p \in \mathbb{Z}_2[x_1, \cdots, x_n]/\langle x_1^2 - x_1, \cdots, x_n^2 - x_n \rangle$$

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Polynomials as sets

$$p = a_1 \cdot x_1^{\nu_{11}} \cdot \ldots \cdot x_n^{\nu_{1n}} + \ldots + a_{2^n} \cdot x_1^{\nu_{2^n 1}} \cdot \ldots \cdot x_n^{\nu_{2^n n}}$$

$$= \sum_{s \in S_p} \left(\prod_{x_\nu \in s} x_\nu \right),$$

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$$= \sum_{s \in S_p} \left(\prod_{x_\nu \in s} x_\nu \right),$$

$$\text{with } S_p = \{\{x_{i_1}, \cdots, x_{i_{n_1}}\}, \cdots, \{x_{i_m}, \cdots, x_{i_{nm}}\}\}$$

$$\subseteq \mathsf{PowerSet}(x_1, \cdots, x_n)$$

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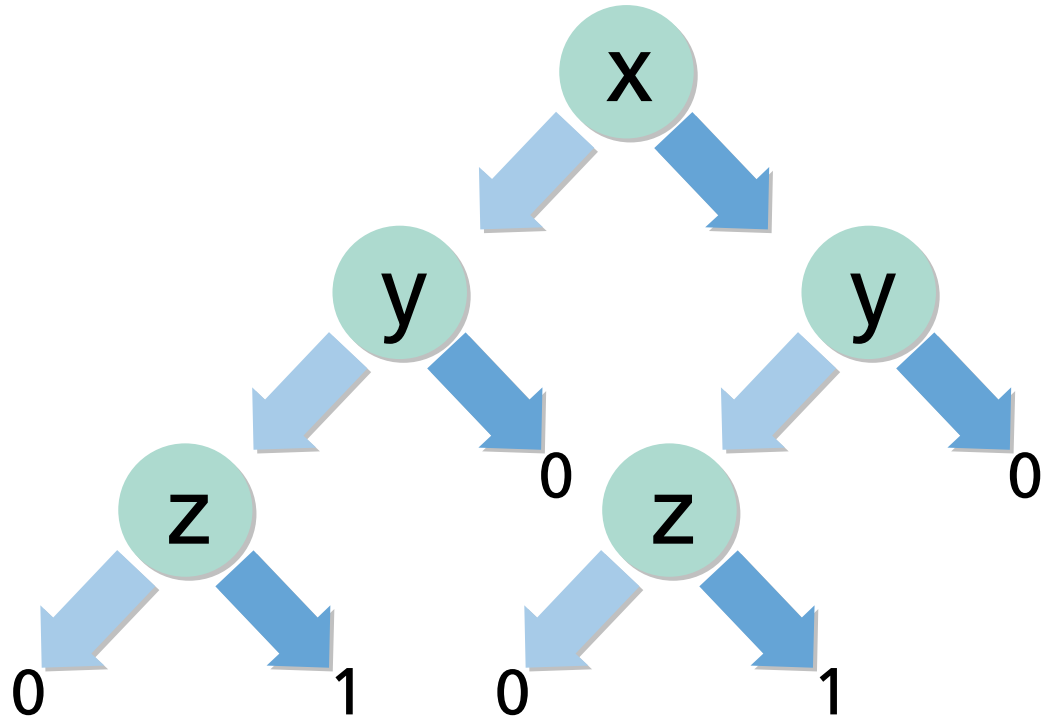
Zero-suppressed Binary Decision Diagrams

Binary Decision Diagram

Rooted, directed, acyclic graph, terminal nodes $\{0, 1\}$, decision nodes ($\widehat{=}$ Boolean variables)



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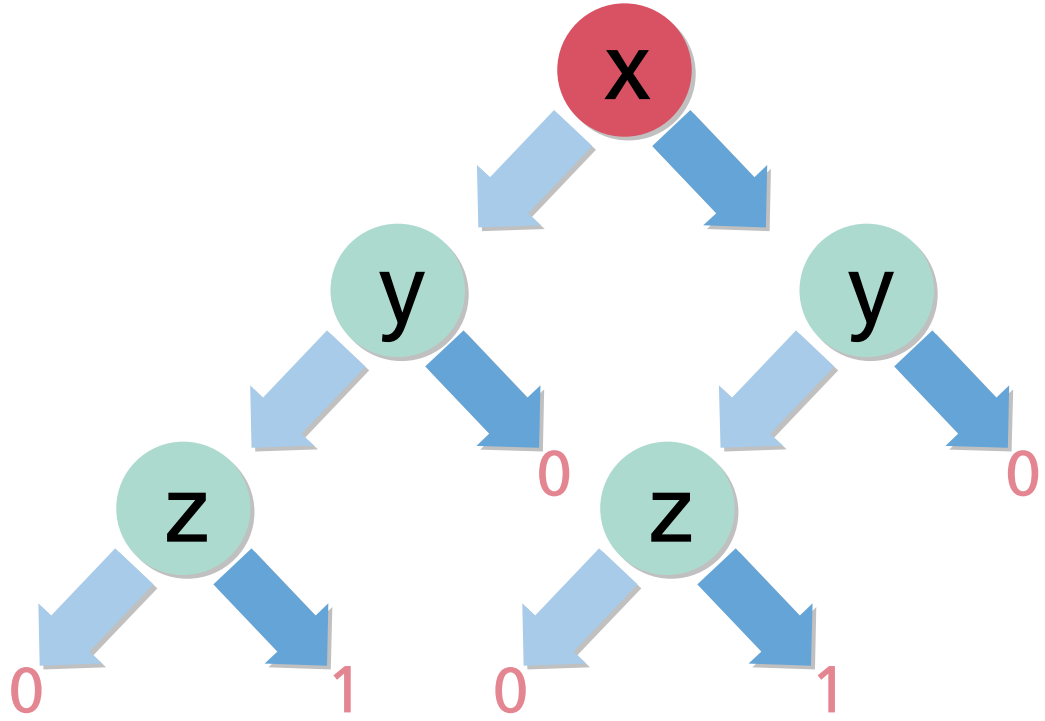

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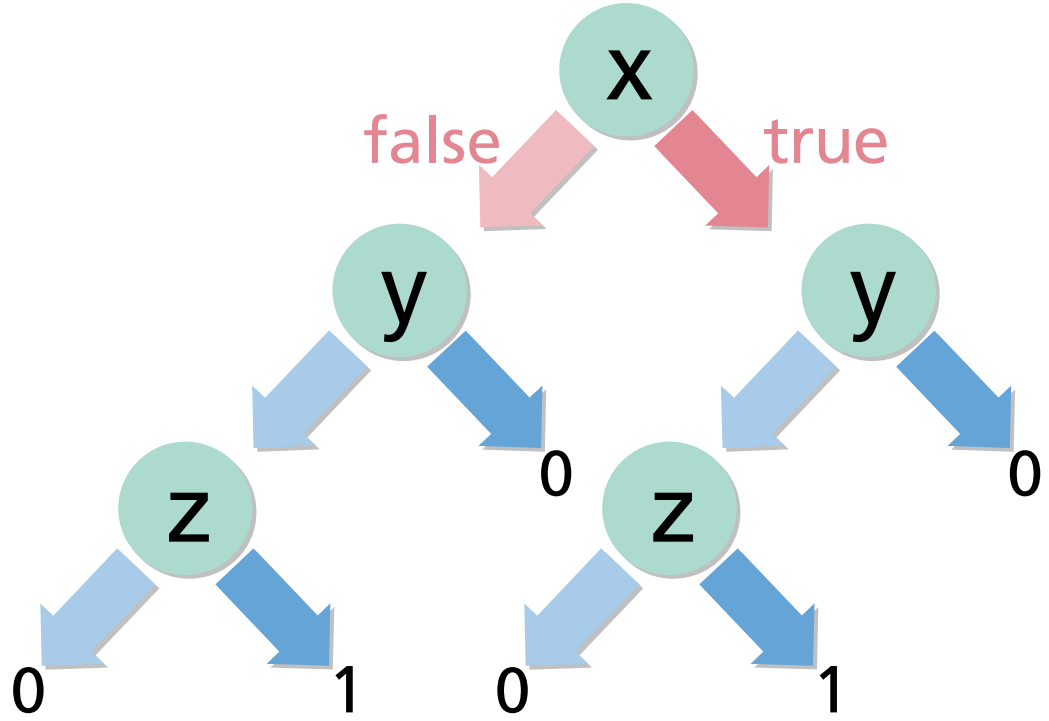

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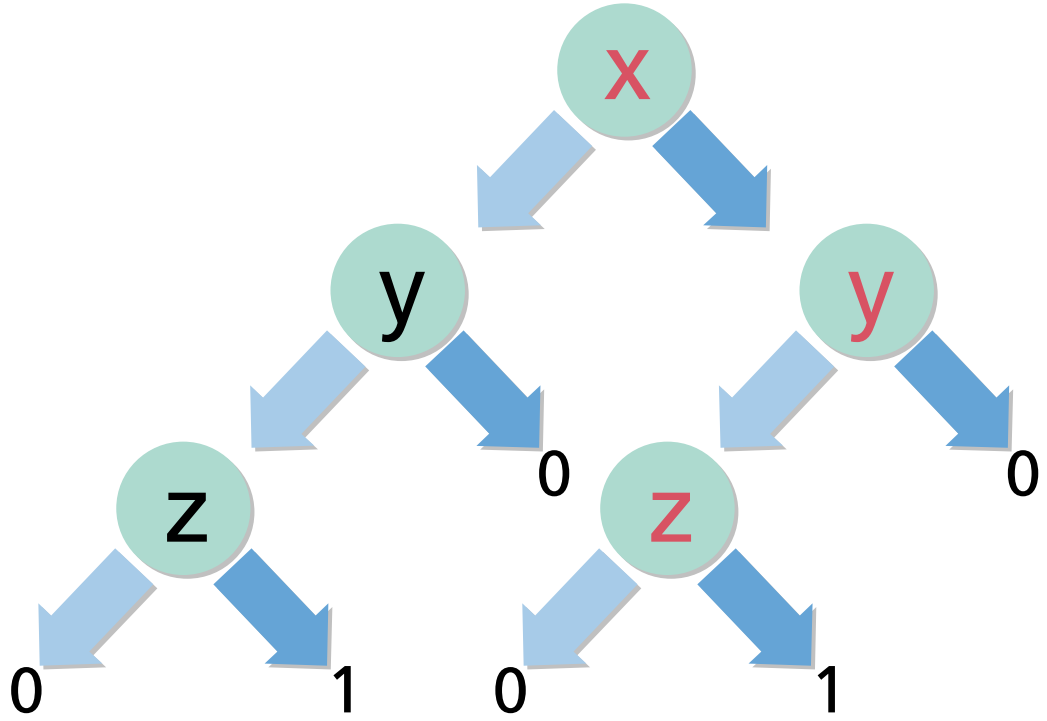

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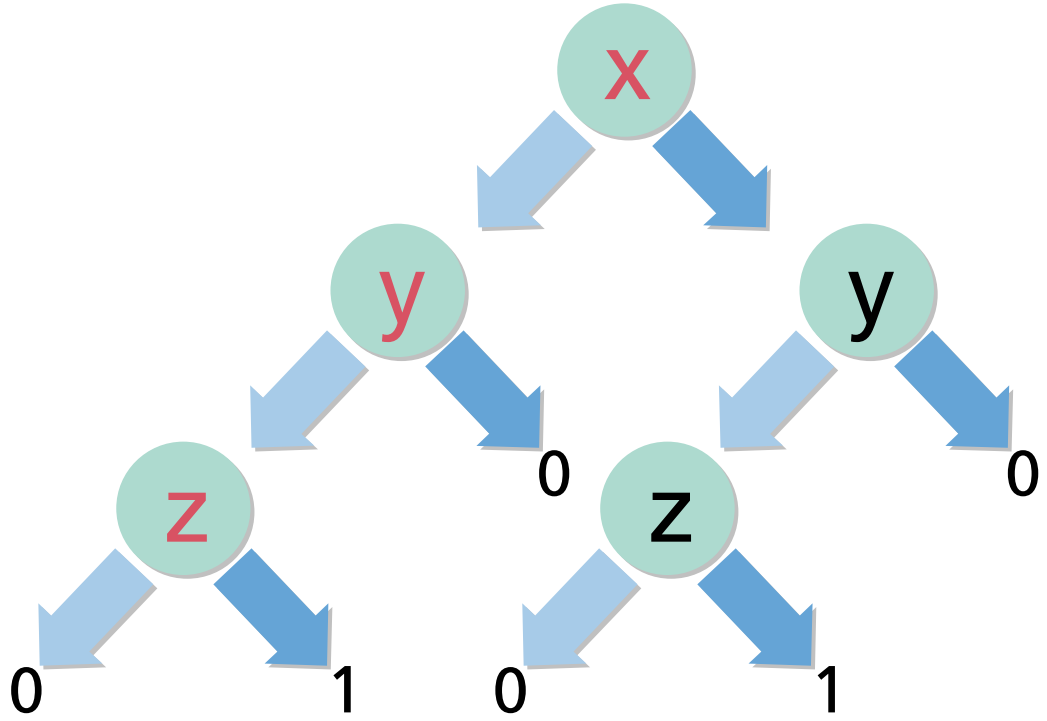

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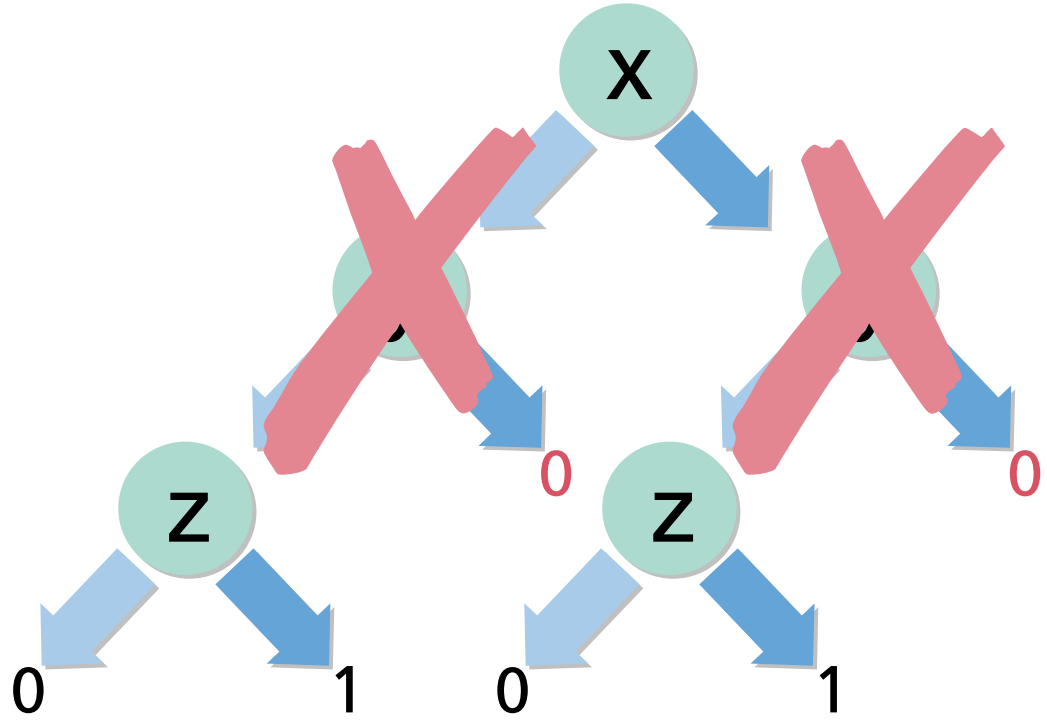
Node eliminated $\iff$ then $\rightarrow 0$



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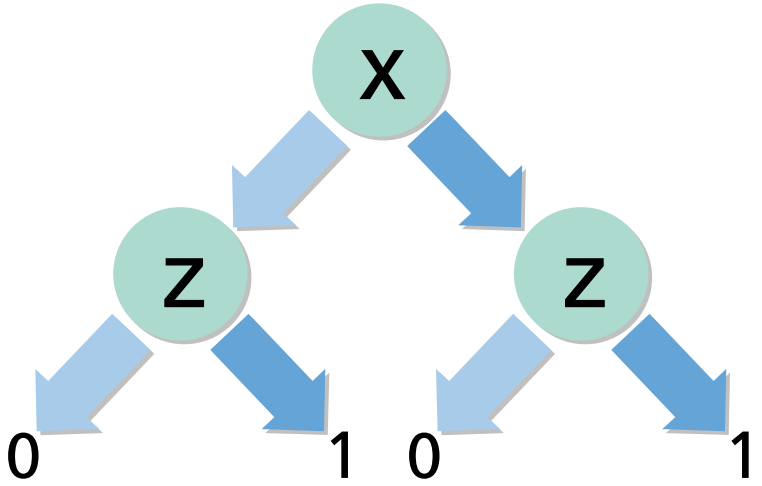

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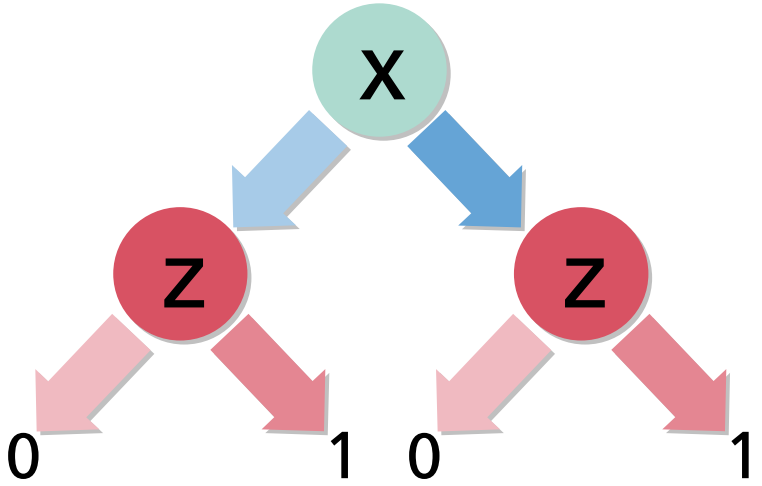

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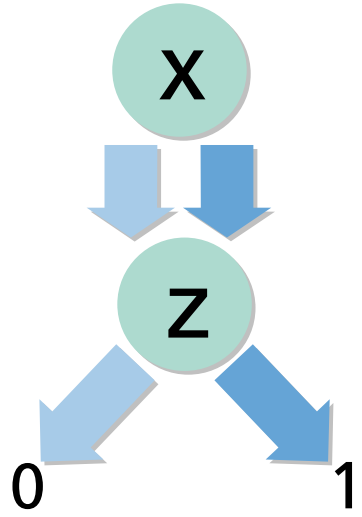

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Advantages of ZDDs

Idea

ZDDs store term structure (**not** the Boolean function behind)

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Idea

ZDDs store term structure (**not** the Boolean function behind)

Reasons

- Compact data structure
- Suitable for sparse sets of subsets
- Polynomial structure recognizable
  - Valid paths $\widehat{=}$ polynomial terms
  - Natural path sequence $\widehat{=}$ lexicographical term ordering

Polynomial Arithmetic

Boolean polynomial operations Correspond to set operations

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Polynomial Arithmetic

Boolean polynomial operations — Correspond to set operations

Example

$$(x + x\,y) + (x\,y + z) = x + 2\,x\,y + z \equiv x + z \iff$$

$$(S_1 \cup S_2) \backslash (S_1 \cap S_2) = \{\{x\}, \{z\}\}$$
(with $S_1 = \{\{x\}, \{x, y\}\}, S_2 = \{\{x, y\}, \{z\}\}$)

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$$x \cdot (y + z) = x\,y + y\,z \iff$$

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Likewise (but more complicated) — Factors/multiples of monomials, degree of a polynomial...

ZDD Implementation — Free C/C++ Library Cudd: Fabio Somenzi (University of Colorado)

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Caching and Recursion

ZDD normalform

Unique diagram root nodes

$a = b \iff$ rootnode(a) is rootnode(b)

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Advantage	Speed-up recursive procedures

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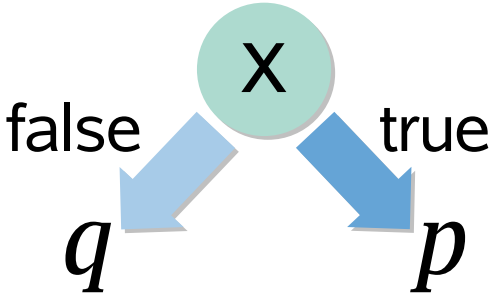
Caching and Recursion

Example: lead(f)



$$f = q + x \cdot p$$

First (lexicographical) term t in f with $\deg t = \deg f$.

Input: f Boolean polynomial **Output:** $t = \mathrm{lead}(f)$ (deg-lex)

if $f \in \{0, 1\}$ **then**
  set $t := 1$
else if isCached(lead, f) **then**
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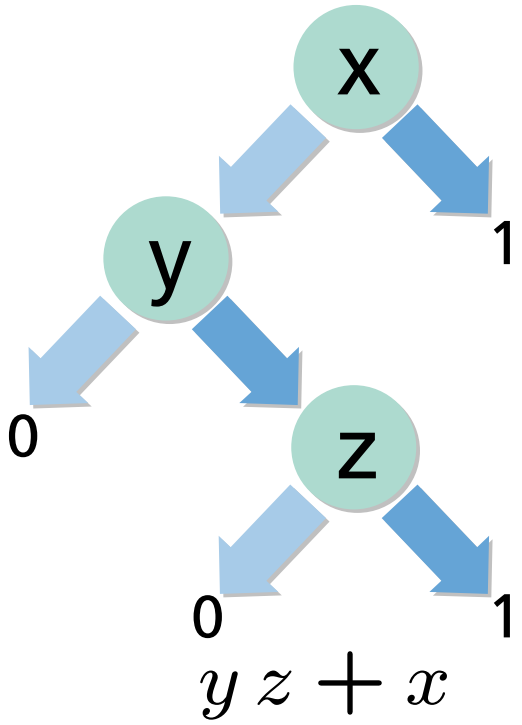

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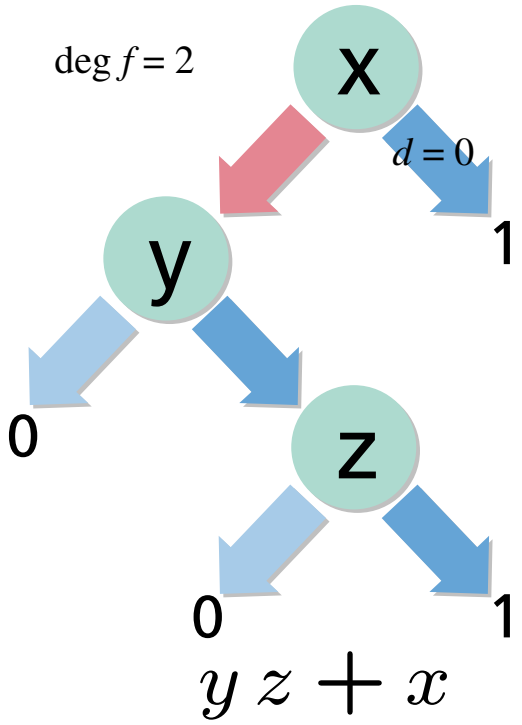

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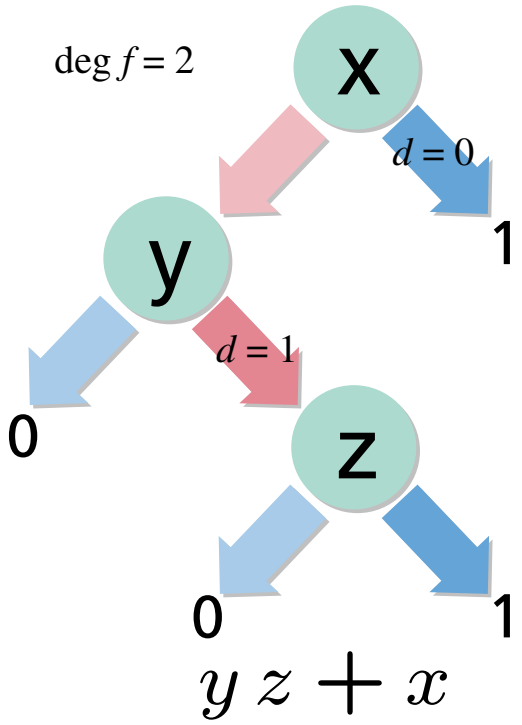

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From ZDDs to PolyBoRi

C++-Library

High-level data types for Boolean polynomials, monomials, exponent vectors, and underlying rings
Implements polynomial operations and basic functionality

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Singular Interface	Prototype

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PolyBoRi's Python Interface

Python interface allows for

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Rapid Prototyping
- Many algorithms existed in python first
- Sophisticated and easy extendable strategies for Gröbner base computation

Formal Verification

Boolean Polynomials and Logical Expressions

- Propositional logic $\longrightarrow$ Boolean polynomial
- Each formula from propositional logic can be stated using Or, Not, True, False
- Choose a mapping: True $\longrightarrow 0$, False $\longrightarrow 1$
  $\mathrm{Or}(x, y) \longrightarrow x \cdot y$, $\mathrm{Not}(x) \longrightarrow 1 + x$

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Property Checking

- Digital system given as set R of Boolean polynomials
- Property p (Boolean expression) to be verified w. r. t. R
- Check via normal form computation: $\mathrm{NF}(p, R) = 0$?

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Representation of Adder Blocks with Carry Bit

Variables

$s(0), \ldots, s(n-1)$ (sums), $c(0), \ldots, c(n-1)$ (carry bits),
$a(0), \ldots, a(n-1), b(0), \ldots, b(n-1)$ (inputs)

Topological Variable Order

$s(n-1), c(n-1), a(n-1), b(n-1),$
$s(n-2), c(n-2), a(n-2), b(n-2), \ldots$

Outputs before inputs, a, b reverse alternating (fast ZDD handling)

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Equations

$c(i) + carry_i$
$s(i) + a(i) + b(i) + carry_i$

with $carry_{-1} = 0$
and $carry_i = a(i) \cdot (b(i) + carry_{i-1}) + b(i) \cdot carry_{i-1}$

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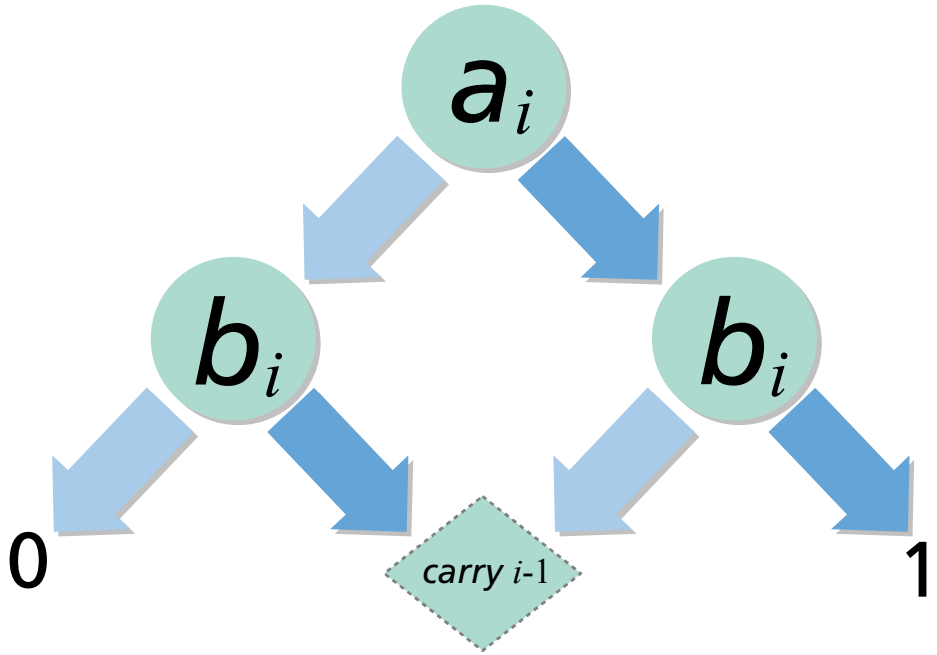
Carry-Bit Graph



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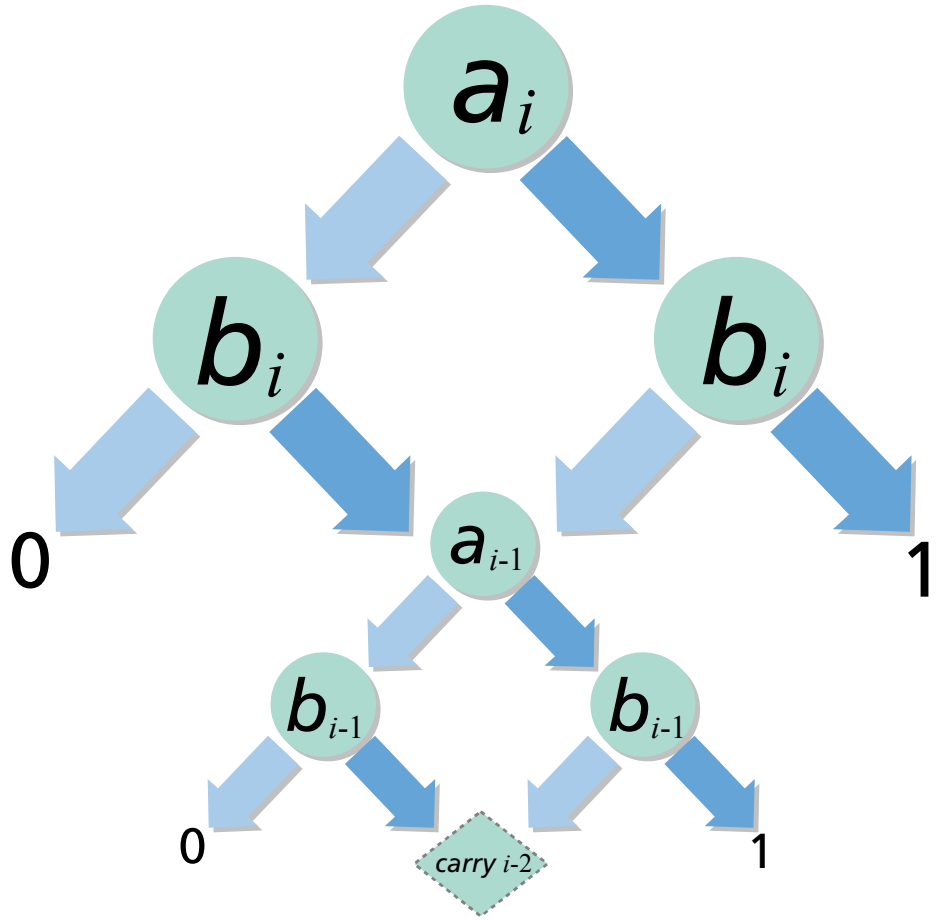
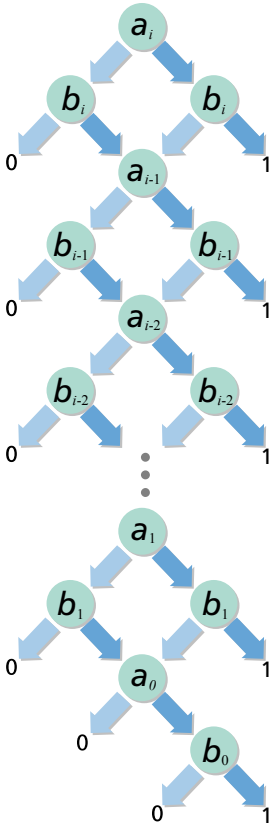


Carry-Bit Graph



i-th bit	terms	nodes
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2	3	5
3	7	8
4	15	11
5	31	14
6	63	17
7	127	20
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9	511	26
10	1023	29
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n	$2^n - 1$	$3\,n - 1$

- Sophisticated ordering of variables
- Linear growth of ZDDs (plaited structure)
- Fast generation and arithmetic operations

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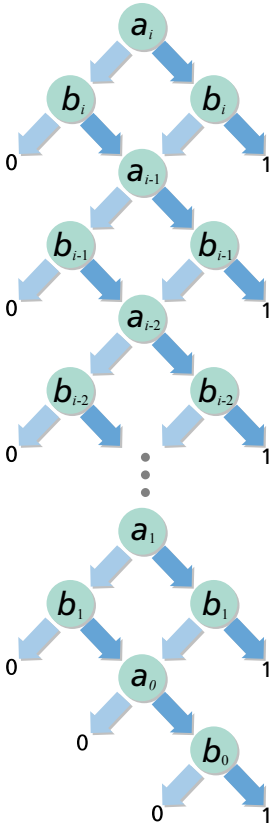

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Demo: $n = 4096$

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Application: Abstract 4096-Bit Adder and Logic

```
adder_bits=4096
adder_block=AdderBlock(sums="s",carries="c",
                       input1="a",input2="b",
                       adder_bits=adder_bits,start_index=1)

declare_ring([Block("x",100),adder_block,Block("y",100)])
ideal=[
    y(0)+y(1)+1,
    a(adder_bits)+1,b(adder_bits)+y(0)+y(1),
    x(0)+(c(adder_bits))*x(1)
]
adder_block.implement(ideal)

claims=[
  c(adder_bits)+1,
  if_then([x(1)],[x(0)]),
  if_then([a(1),b(1)],[c(1),s(1)]),
  if_then([a(1)+1],[c(1)+b(1),s(1)+b(1)+1]),
  s(adder_bits)+c(adder_bits-1),
  x(33)
]
```

Counter Example for Failed Claims

- A claim c fails $\iff$ $q := \mathsf{NF}(c, R) \neq 0$
- A Boolean polynomial is 0 $\iff$ it is 0 as a function

Algorithm

Boolean polynomial q reduced w. r. t. R, R red. Gröbner basis

$\mathsf{Variables}(q) \subseteq V := \{x_1, .., x_n\} \backslash \{\text{Leading Terms}(R)\}$

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Now we have to make it compatible with relations in R:

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Step 2 For every $x_i \in \mathsf{lead}(R)$, we have $p \in R$, with $x_i = \mathsf{lead}(x)$ and $\mathsf{tail}(p) \in K[V]$

Redefine $v_i := \mathsf{tail}(p)(v_1, \ldots, v_n)$

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Theorem

$v_1, \ldots, v_n$ provides a valid counter example

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Algorithm: Find a One of a Boolean Polynomial

Input: Boolean polynomial $p \neq 0$

Out: $\{x_i \to v_i\}_i : p(v_1, \ldots, v_k) = 1$

if $p = 1$ **then**
  return $\emptyset$ // empty tuple
end if
Set $r :=$ rootnode(p)
Set $x :=$ Variable corresponding to r
if elseBranch(r) is 0 **then**
  Set $v = 1$
  Set $q =$ thenBranch(r) // plug in 1
else
  Set $v = 0$
  Set $q =$ elseBranch(r) // plug in 0
end if
return $\{x \to v\} \cup$ find_one(q)

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Summary

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- Optimized setup of arithmetic components (adder)

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