PolyBoRi - A Gröbner Basis Framework for Boolean Polynomials

Michael Brickenstein

Mathematisches Forschungsinstitut Oberwolfach

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Oberseminar im Wintersemester 2007/2008
Kryptographie und Computeralgebra, TU Darmstadt, 31. 1. 2008

*Presentation

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Basic Data

PolyBoRi	***Poly***nomials over ***Bo***olean ***Ri***ngs
Website	`http://polybori.sourceforge.net`
DFG Project	"Development, implementation and application of mathematical-algebraic algorithms for formal verification of digital systems with arithmetic blocks"
Fraunhofer ITWM	Alexander Dreyer (Department Adaptive Systems)
University of Kaiserslautern	Algebra, Geometry and Computer Algebra Group (Prof. Greuel, Department of Mathematics)
Doctoral thesis	Michael Brickenstein (Mathematisches Forschungsinstitut Oberwolfach)

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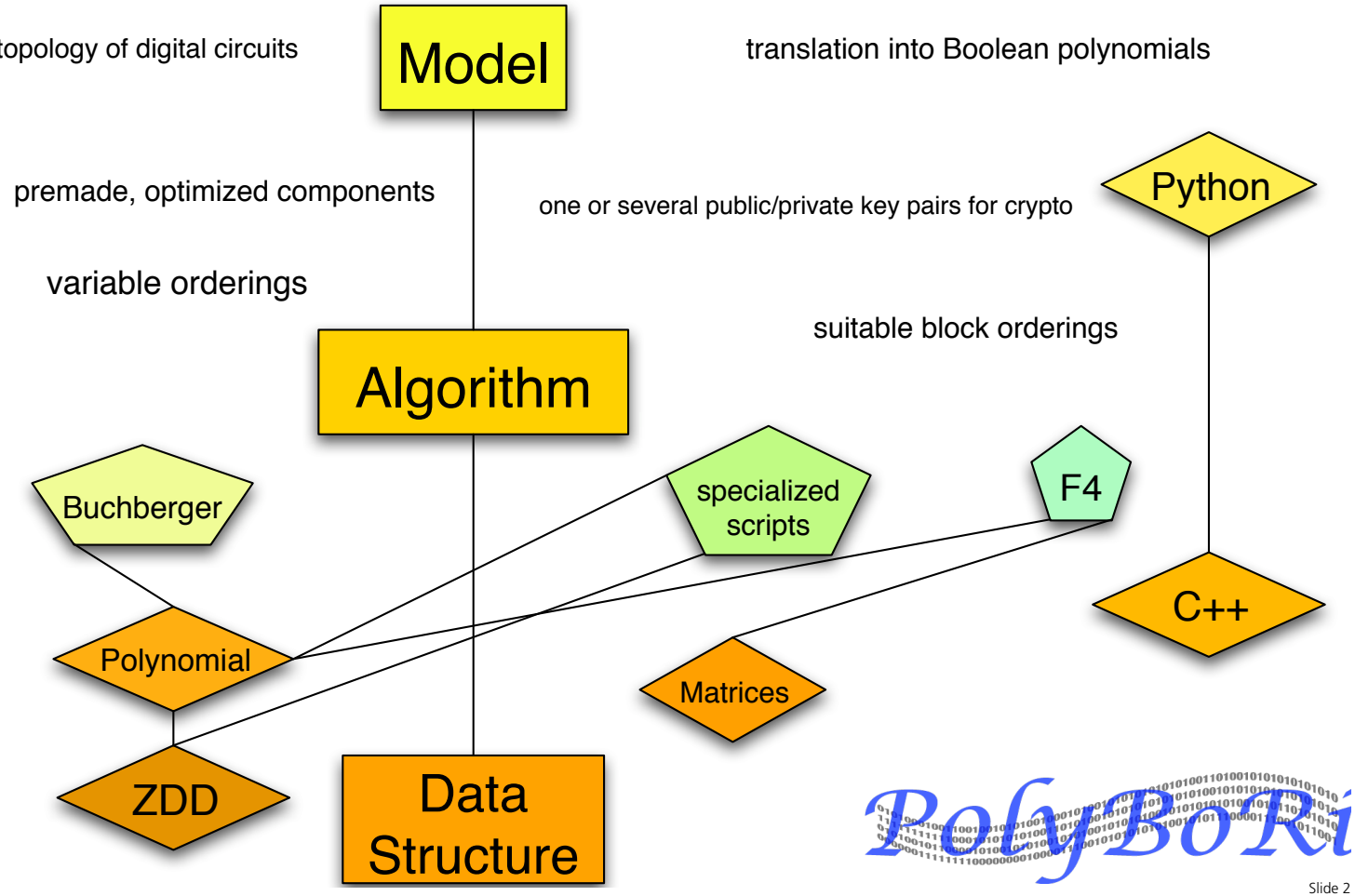
Overview

Optimization on many levels



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Translation from $\mathrm{GF}(2^r)$ to $\mathrm{GF}(2)$

Identification of fields

$\mathrm{GF}(2^r) \leftrightarrow \mathrm{GF}(2)[a]/\langle m \rangle =: F$,
irreducible $m \in \mathrm{GF}(2)[a]$ with $\deg m = r$

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Identification of rings

$$\mathrm{GF}(2^r)[x_1, \ldots x_n] \leftrightarrow \mathrm{GF}(2)[a][x_1, \ldots, x_n]/\langle m \rangle =: P$$

Monomorphism of F-algebras

$$\phi : P \to P_2, \quad x_i \mapsto \sum_{j=0}^{r-1} y_{i,j} \cdot a^j$$

$$P_2 = F[y_{1,0}, \ldots, y_{1,r-1}, y_{2,0}, \ldots, y_{2,r-1}, \ldots, y_{n,0}, \ldots, y_{n,r-1}]$$

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Canonical components f_i

Let $f \in P \Rightarrow \exists f_i \in \mathrm{GF}(2)[y_{1,0}, \ldots, y_{1,r-1}, y_{2,0}, \ldots, y_{n,r-1}] =: Q$
s. th. $\phi(f) = \sum_{i=0}^{r-1} a^i \cdot f_i$

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Translation from $\mathrm{GF}(2^r)$ to $\mathrm{GF}(2)$

Canonical components $\phi(f) = \sum_{i=0}^{r-1} a^i \cdot f_i$

Map to canonical components $\tau : P \to \mathsf{PowerSet}(Q), \quad f \mapsto \{f_0, \ldots, f_{r-1}\}$

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Correspondence $G = \{g_1, \ldots, g_m\} \subset P \leftrightarrow H = \cup_{g \in G} \tau(f) \subset Q$

Isomorphism of $\mathrm{GF}(2)$-vector spaces $\lambda : F^n \to \mathrm{GF}(2)^{r \cdot n}$, $a^j \cdot e_i \mapsto E_{(i-1)\cdot r + j + 1}$ $\quad (F = \mathrm{GF}(2)[a]/\langle m \rangle \cong \mathrm{GF}(2^r))$

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Isomorphism of $\mathrm{GF}(2)$-vector spaces

$\lambda\colon\ F^n \to \mathrm{GF}(2)^{r \cdot n}$ $\qquad (F = \mathrm{GF}(2)[a]/\langle m \rangle \mathrel{\widehat{=}} \mathrm{GF}(2^r))$

$a^j \cdot e_i \mapsto E_{(i-1)\cdot r + j + 1}$

Theorem $\lambda(\mathsf{V}(G \cup \mathrm{FP}_P)) = \mathsf{V}(H \cup \mathrm{FP}_Q)$

Field polynomials

$\mathrm{FP}_P = \{x_1^{2^r} - x_1, \ldots, x_n^{2^r} - x_n\},$
$\mathrm{FP}_Q = \{y_{i,j}^2 - y_{i,j} \mid 1 \le i \le n, 0 \le j < r\}$

Describe vanishing points over $\mathrm{GF}(2^r)$ and $\mathrm{GF}(2)$, resp.

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Translation from $\mathrm{GF}(2^r)$ to $\mathrm{GF}(2)$

Disadvantages

- More equations (r times as much)

Advantages

- Simpler base field $\mathrm{GF}(2)$
- Lower degree per variable (bound to 1, if FP_Q is omitted)
- Boolean polynomial approach applicable (PolyBoRi)

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Digital systems verification and Boolean Polynomials

Interprete Boolean expressions as polynomials over $\mathbb{Z}_2$

logical operations $\longrightarrow$ arithmetical operations and $\{\mathsf{true}, \mathsf{false}\} \longrightarrow \{0, 1\}$

$$p \in \mathbb{Z}_2[x_1, \cdots, x_n]/\langle x_1^2 - x_1, \cdots, x_n^2 - x_n \rangle$$

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Polynomials as sets

$$p = a_1 \cdot x_1^{\nu_{11}} \cdot \ldots \cdot x_n^{\nu_{1n}} + \ldots + a_{2^n} \cdot x_1^{\nu_{2^n 1}} \cdot \ldots \cdot x_n^{\nu_{2^n n}}$$

$$= \sum_{s \in S_p} \left(\prod_{x_\nu \in s} x_\nu \right),$$

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$$= \sum_{s \in S_p} \left(\prod_{x_\nu \in s} x_\nu \right),$$

with $S_p = \{\{x_{i_1}, \cdots, x_{i_{n_1}}\}, \cdots, \{x_{i_m}, \cdots, x_{i_{nm}}\}\}$

$$\subseteq \mathsf{PowerSet}(x_1, \cdots, x_n)$$

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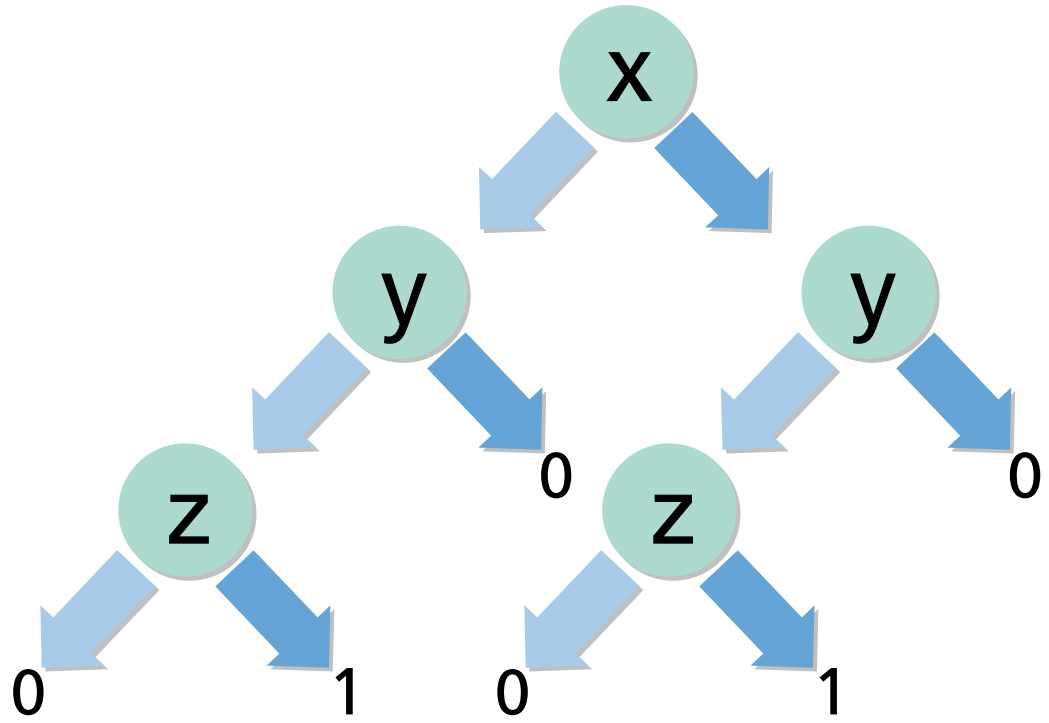
Zero-suppressed Binary Decision Diagrams

Binary Decision Diagram

Rooted, directed, acyclic graph, terminal nodes $\{0, 1\}$, decision nodes ($\widehat{=}$ Boolean variables)



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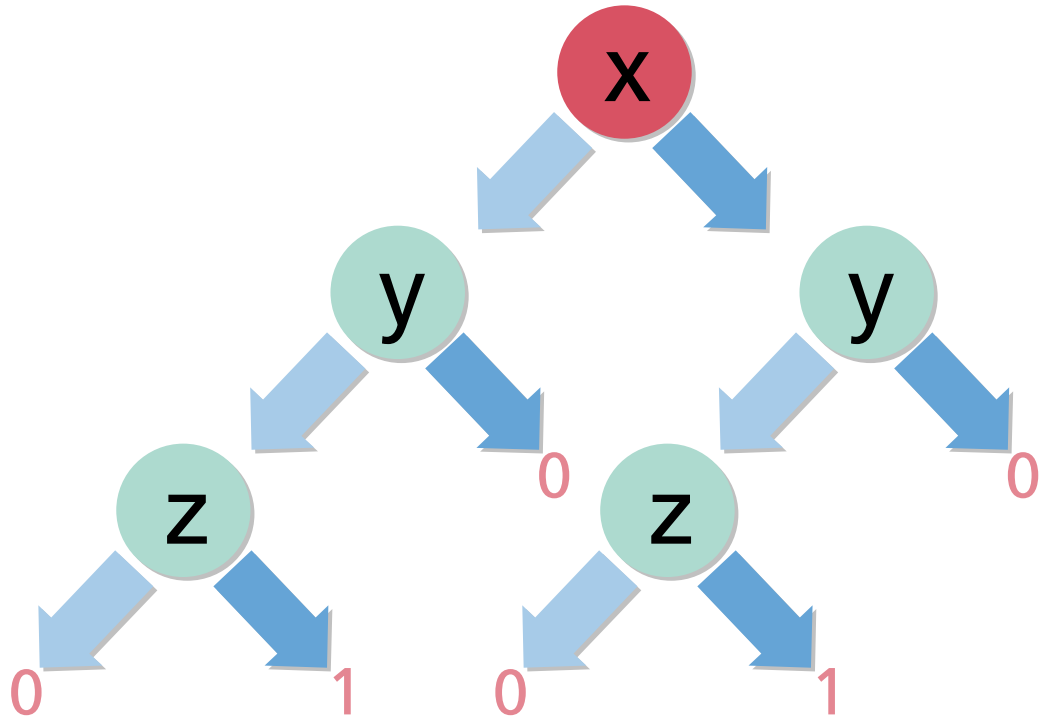

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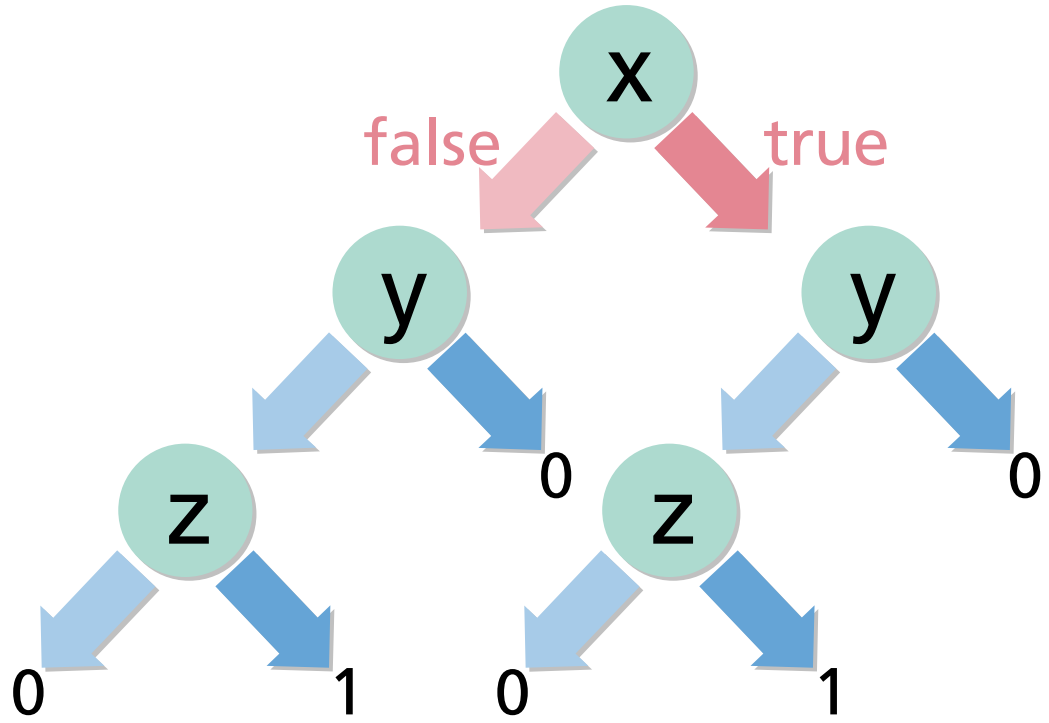
$\widehat{=}$ assign variable to true/false



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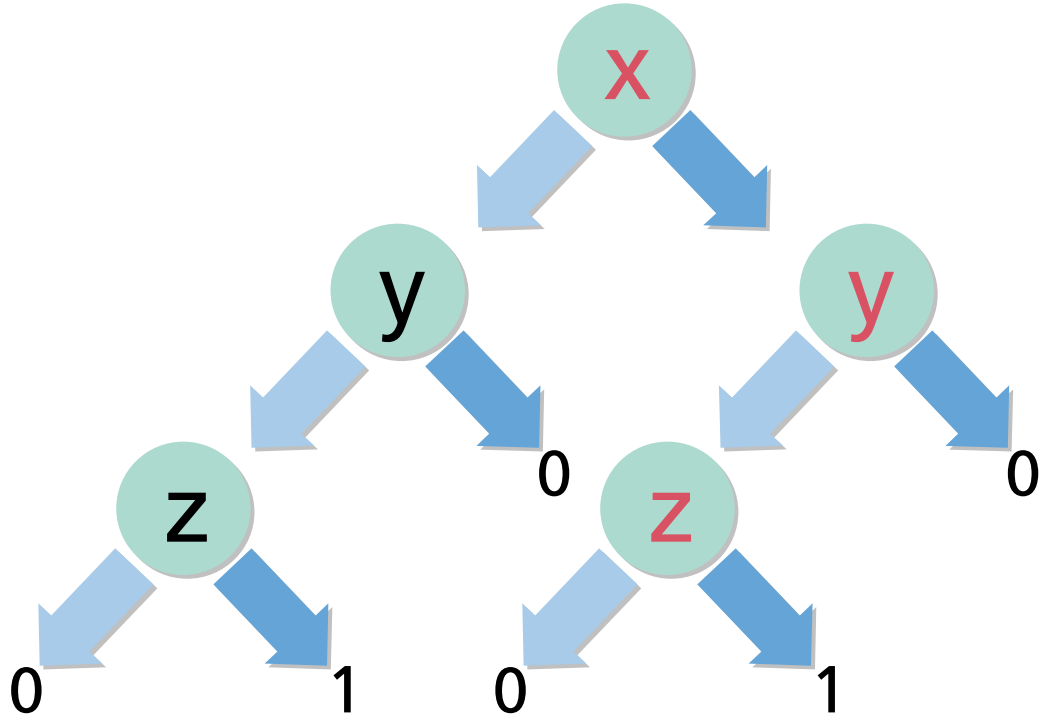
»Ordered« if the variable order is constant over all paths



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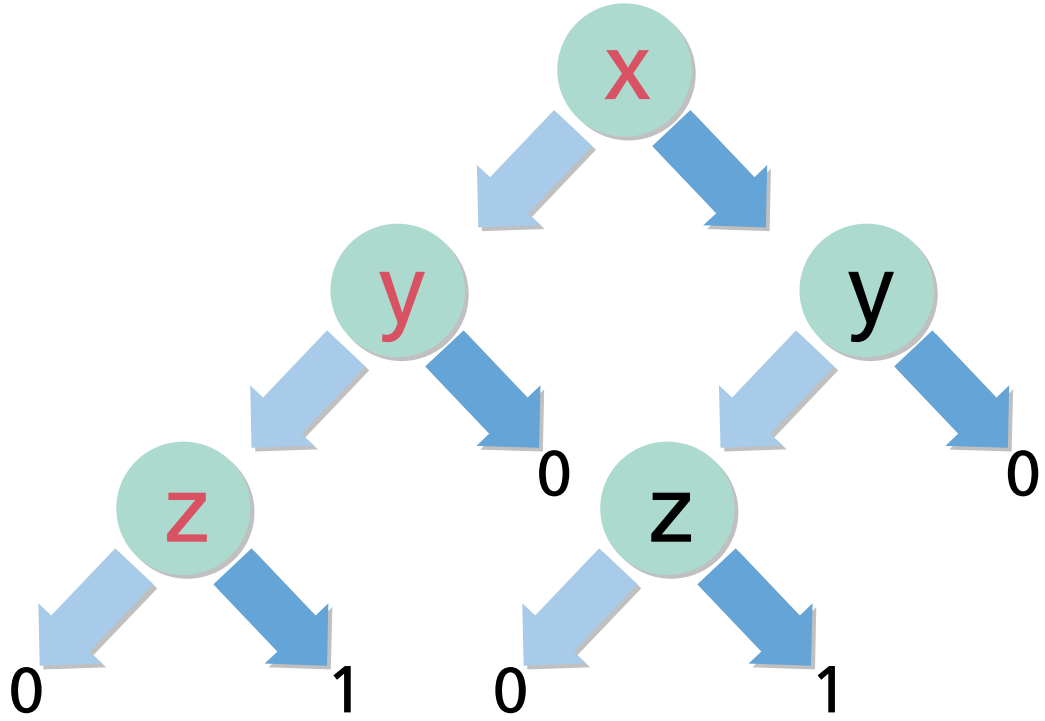

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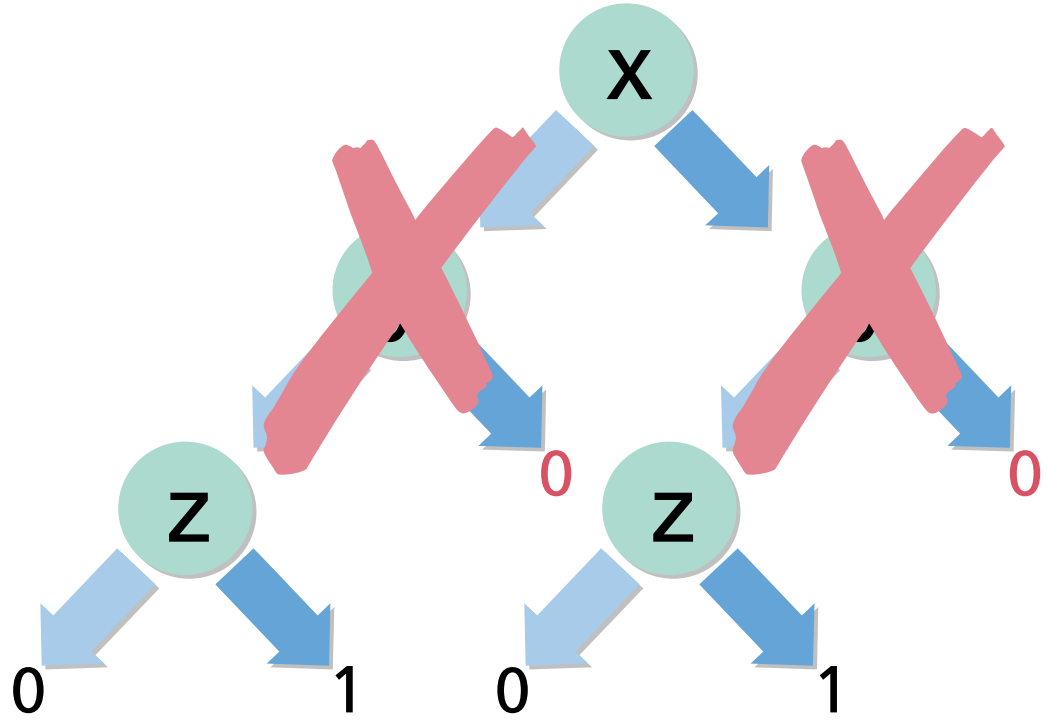
Zero-suppressed BDD (ZDD)

Node eliminated $\iff$ then $\rightarrow 0$



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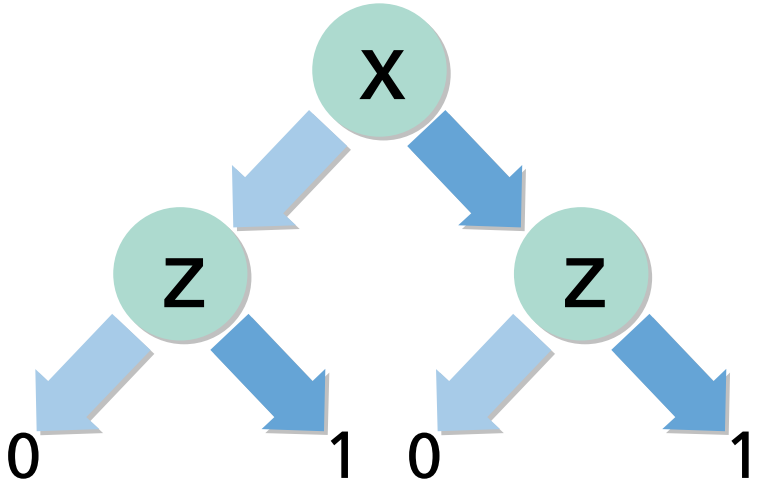

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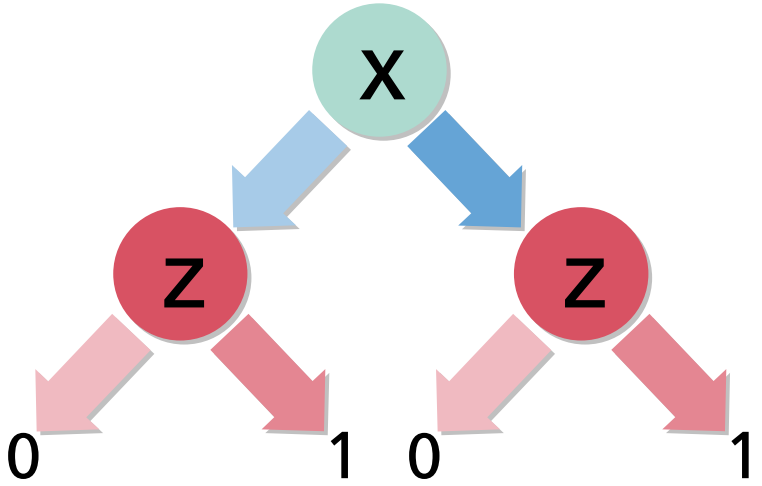

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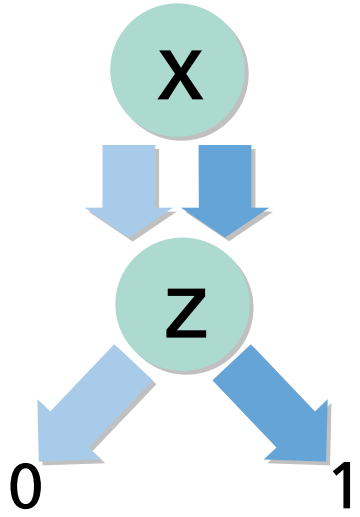

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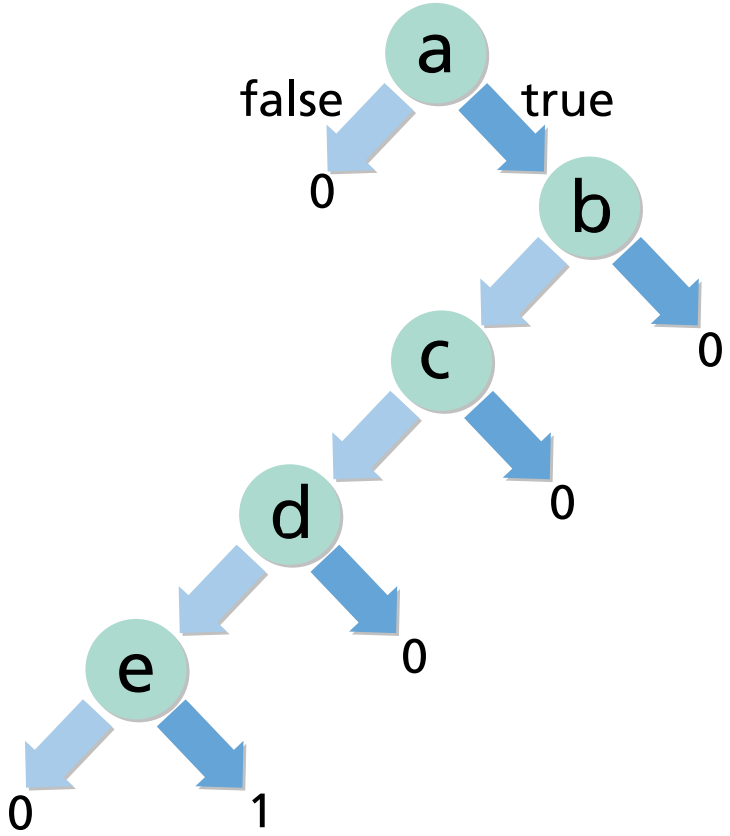
ZDDs for Sparse Sets

Example: $\{\{a, e\}\}$



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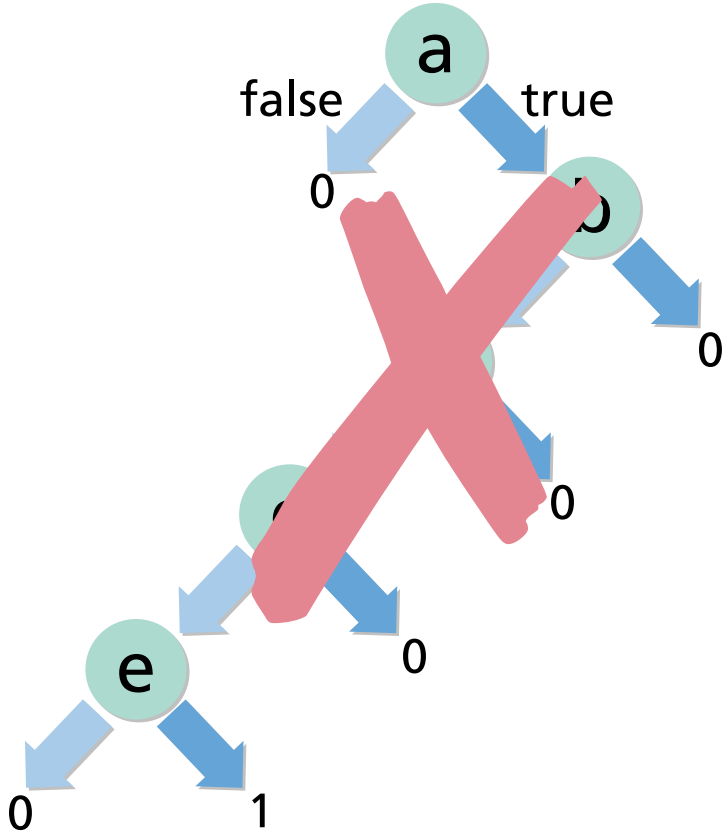

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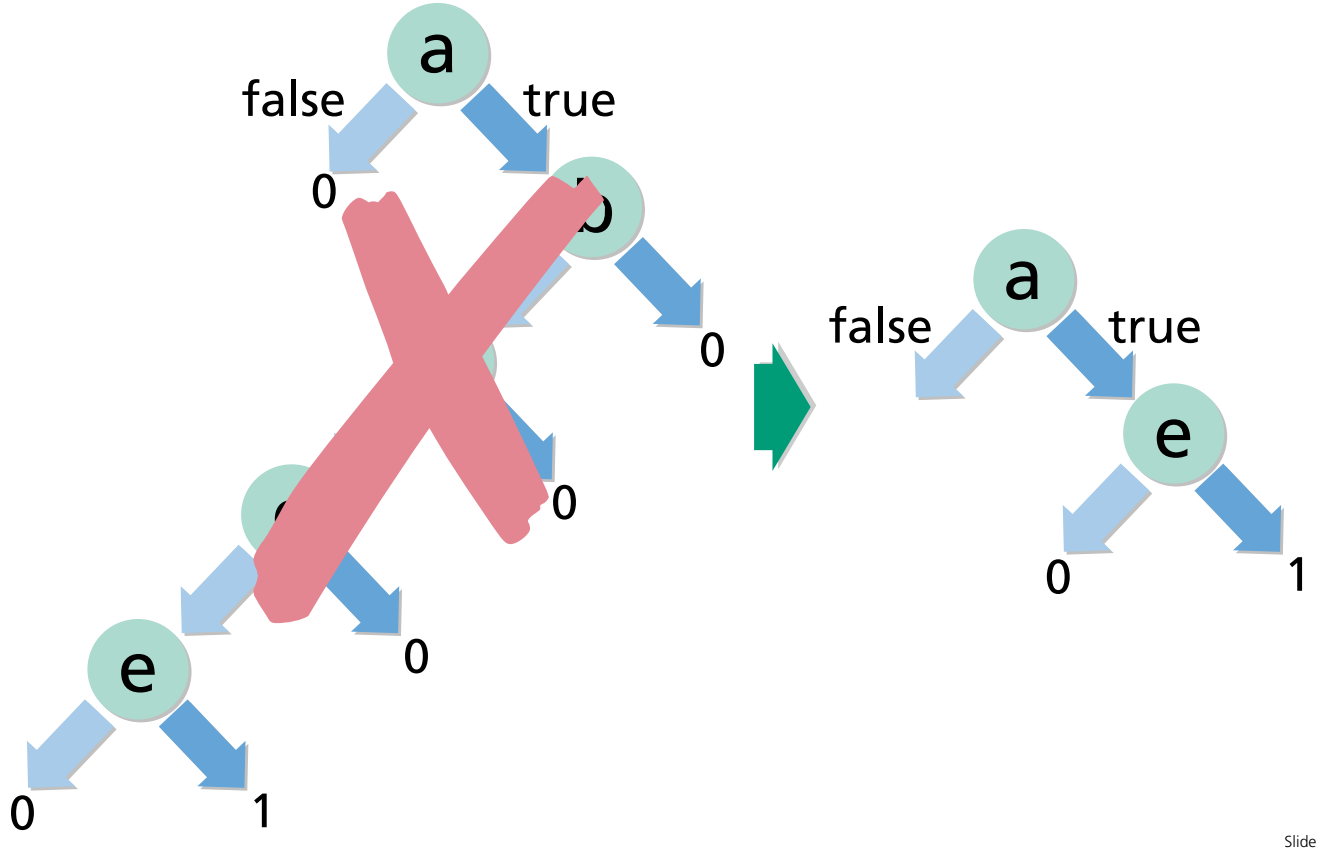

Advantages of ZDDs

Idea

ZDDs store term structure (**not** the Boolean function behind)

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  - Paths $\widehat{=}$ polynomial terms
  - Natural path sequence $\widehat{=}$ lexicographical term ordering

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  - Paths $\widehat{=}$ polynomial terms
  - Natural path sequence $\widehat{=}$ lexicographical term ordering
- Unique diagram nodes (less memory, reference counting)
- Caching of operations

Boolean Polynomials as ZDDs

Example $x\,y + x + z \mathrel{\widehat{=}} \{\{x, y\}, \{x\}, \{z\}\}$

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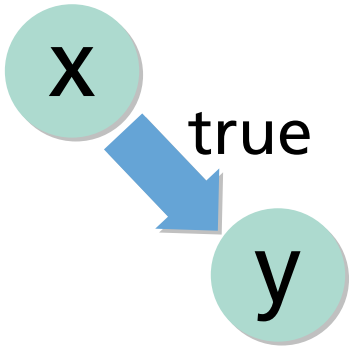

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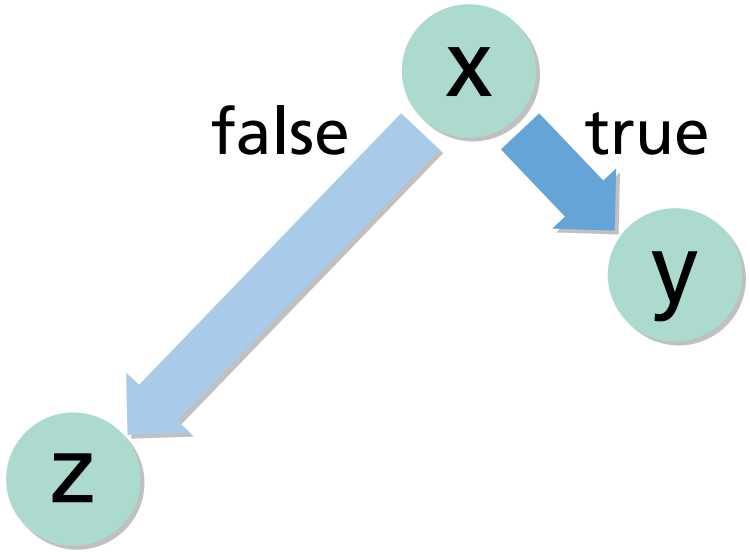

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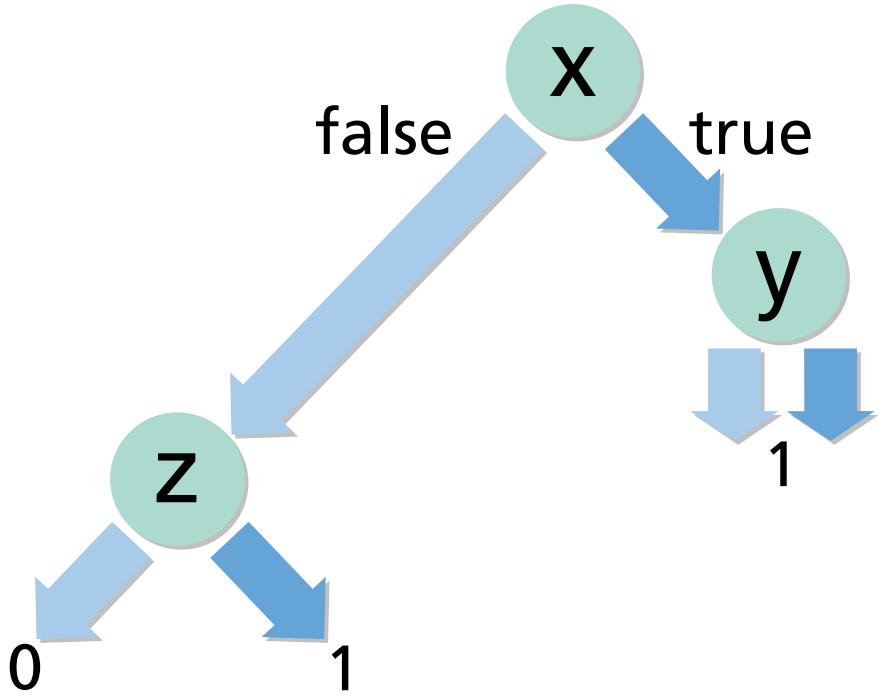

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Polynomial Arithmetic

Boolean polynomial operations Correspond to set operations

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Polynomial Arithmetic

Boolean polynomial operations — Correspond to set operations

Example

$$(x + x\,y) + (x\,y + z) = x + 2\,x\,y + z \equiv x + z \iff$$

$$(S_1 \cup S_2)\backslash(S_1 \cap S_2) = \{\{x\}, \{z\}\}$$
(with $S_1 = \{\{x\}, \{x, y\}\}, S_2 = \{\{x, y\}, \{z\}\}$)

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$$x \cdot (y + z) = x\,y + y\,z \iff$$

$$\{\{x\} \cup \{y\}, \{x\} \cup \{z\}\} = \{\{x, y\}, \{x, z\}\}$$

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$$x \cdot (y + z) = x\,y + y\,z \iff$$

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Likewise (but more complicated) — Factors/multiples of monomials, degree of a polynomial...

ZDD Implementation — Free C/C++ Library Cudd: Fabio Somenzi (University of Colorado)

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From ZDDs to PolyBoRi

C++-Library

High-level data types for Boolean polynomials, monomials, exponent vectors, and underlying rings
Implements polynomial operations and basic functionality

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Non-trivial Monomial Orderings (run- & compile-time selectable)	Lexicographic, degree-lexicographic, and degree-reverse-lexicographic (ascending variable order) orderings Orderings consisting of blocks of degree-orderings

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Interfaces `www.sagemath.org` `singular.mathematik.uni-kl.de`	SAGE (python code compatible; community-made) Singular (Prototype)

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Data Structures

Polynomial Ring
(tiny excerpt)

Generates base components

Ordering (runtime changeable)

Active ring (shared pointer) based on decision diagram manager

```
class BoolePolyRing {
 public:
  // ...
  BoolePolyRing(size_type nvars, ordercode_type order = lp);

  dd_type variable(idx_type i) const;  // i-th variable
  dd_type zero() const;          // Constant polynomials
  dd_type one() const;

  size_type nVariables() const;

  void changeOrdering(ordercode_type);


 protected:
  manager_ptr pMgr;
  // ...
};
```

Data Structures

Example
(tiny excerpt)

Object-oriented operations

High-level abstractions

Likewise for monomials,
variables, and exponent vector

```
class BoolePolynomial {
 public:
  // ...
  BoolePolynomial();                // Default constructor
  BoolePolynomial(bool_type);       // Construct from 0 or 1

  self& operator+=(const self&); // Arithmetical operations
  self& operator*=(const self&);

  bool operator==(const self&) const; // Logical operations
  bool operator!=(const self&) const;

  monom_type lead() const;          // Get leading term
  size_type deg() const;            // Degree of the polynomial
  monom_type usedVariables() const;
  // ... various term access functions ...
 private:
  dd_type m_dd;                     // Actual decision diagram
};
```

Term-access Functions

Forward iterator over terms

```
const_iterator begin() const;
const_iterator end() const;
```

in natural – lexicographical – term ordering

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Runtime-selected ordering

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```

Compile-time selected ordering

$\langle order \rangle$`_iterator genericBegin(`$\langle order \rangle$`_tag) const;`
$\langle order \rangle$`_iterator genericEnd(`$\langle order \rangle$`_tag) const;`

currently for lexicographic, degree-lexicographic, and degree-reverse-lexicographic (ascending variable order) orderings

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currently for lexicographic, degree-lexicographic, and degree-reverse-lexicographic (ascending variable order) orderings

Iterator-like structure for navigation over ZDD nodes

```
navigator navigation() const;
```

`incrementThen()`, `incrementElse()` replacing `operator++()`

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PolyBoRi's Python Interface

Python interface allows for

- Parsing of complex polynomial systems (Inner-domain specific language)
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Rapid Prototyping
- Many algorithms existed in python first
- Sophisticated and easy extendable strategies for Gröbner base computation

Python Interface (Data Format)

Script level

Easy input and handling of complex polynomial systems

```
declare_ring([
  AlternatingBlock(["c","s"],56,start_index=1,reverse=True),
  AlternatingBlock(["a","b"],8,reverse=True),
],globals())

ideal = [
  a(0)*b(1)*a(1)*b(0)+c(1),
  a(0)*b(1)+a(1)*b(0)+s(1),
  a(1)*b(1)*a(2)*b(0)+c(3)
]

ideal = function(ideal);
```

Several variants for Gröber base computation

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Comparison of Storage Types for Boolean Polynomials

Tune strategies and heuristics to strengths of underlying data structure!

	ZDD	linked list	buckets[1]	factory[2]
leading term	++|0 |– –[3]	++	+	+ (close ZDD)
p1 + p2	++	+	++	0
tail(p)[4]	–	++	+	0
redNF(p,m)	++	0	–	– –
canonicalize	++	+	– –	+
memory use	++[5]	+	–	– –
iteration	0[6]	++	– –	– (see ZDD)
p1 * p2	++[7]	+	– –	+
factorize	++	–	– –	++

[1]) partly reduced container [2]) recursive expression
[3]) lex | special cases | general case [4]) often during reduction with monomial [5]) every substructure stored once
[6]) possible, but ordering-dependent [7]) automatically reduced

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Caching and Recursion

ZDD normalform

Unique diagram root nodes
$a = b \iff$ rootnode(a) is rootnode(b)

Caching and Recursion

ZDD normalform	Unique diagram root nodes $a = b \iff \texttt{rootnode}(a)$ is $\texttt{rootnode}(b)$
Reference counting	Lower memory usage (no deep copies)
Caching of operations	$a \diamond b$ never evaluated twice (also for sub-diagrams)

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ZDD normalform	Unique diagram root nodes $a = b \iff \texttt{rootnode}(a)$ is $\texttt{rootnode}(b)$
Reference counting	Lower memory usage (no deep copies)
Caching of operations	$a \diamond b$ never evaluated twice (also for sub-diagrams)
Advantage	Speed-up recursive procedures

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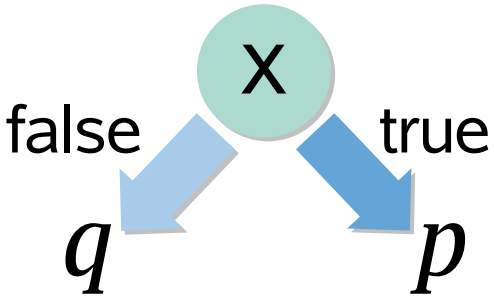
Caching and Recursion

Example: $\deg(f)$



$$f = q + x \cdot p$$

Input: f Boolean polynomial **Output:** $d = \deg(f)$

if $f \in \{0, 1\}$ **then**
  set $d := 0$
else if isCached(deg, f) **then**
  set $d :=$ cache(deg, f)
else
  set $d :=$ max(deg(thenBranch(f)) + 1,
    deg(elseBranch(f)))
  insert cache(deg, f) $:= d$
end if

Caching and Recursion

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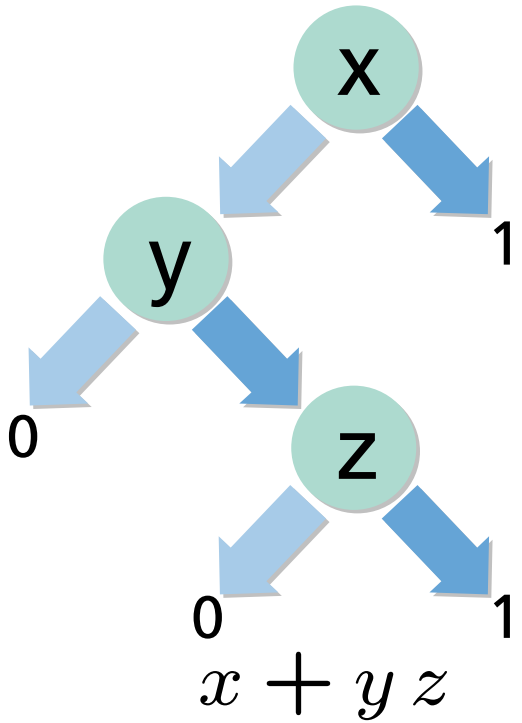

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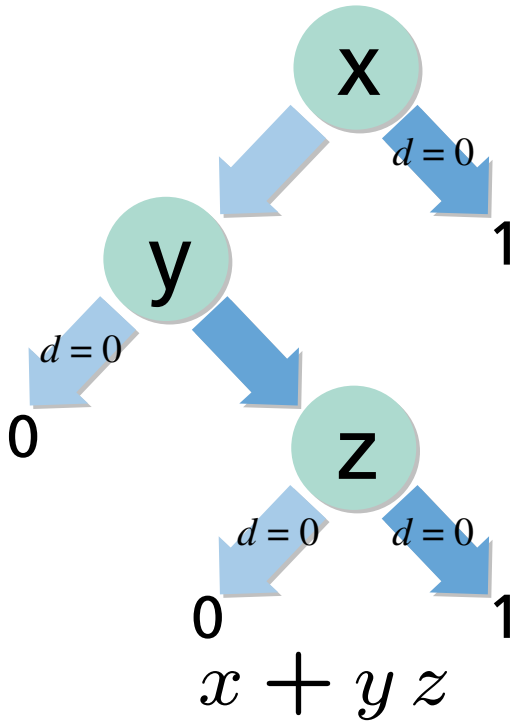

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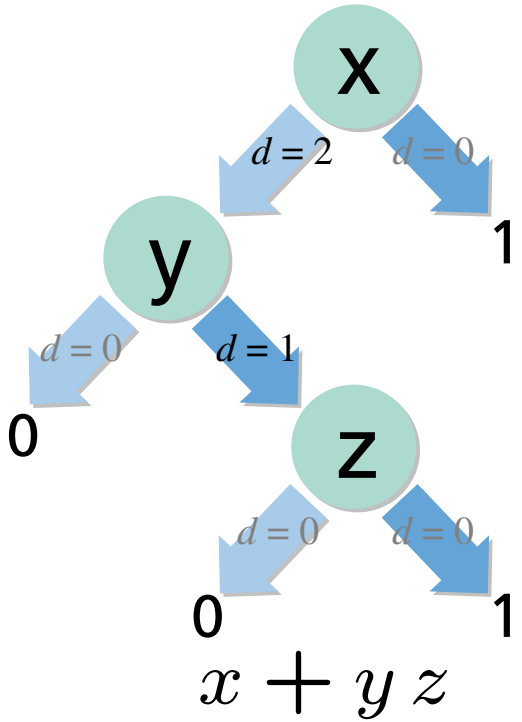

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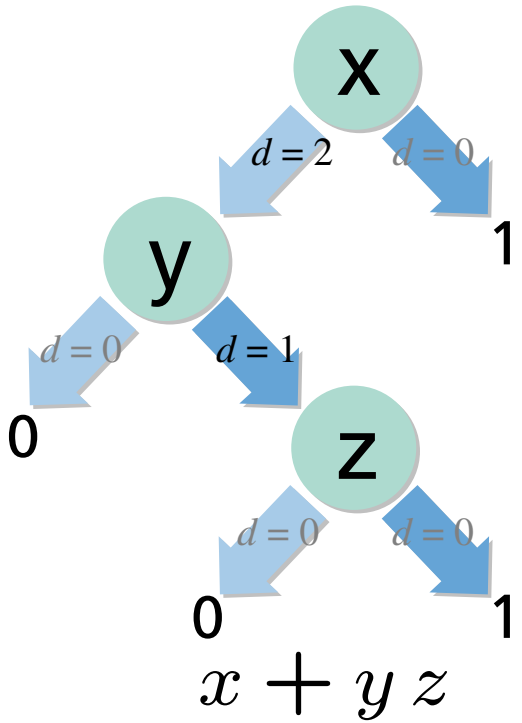

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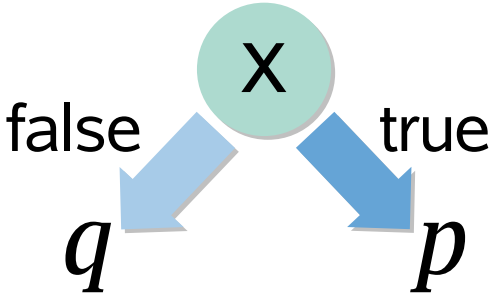
- Reuse for polynomials with likewise structure.
- Fast access to intermediate results by other functions.

Caching and Recursion

Example: lead(f)



$$f = q + x \cdot p$$

First (lexicographical) term t in f with $\deg t = \deg f$.

Input: f Boolean polynomial **Output:** $t = \mathrm{lead}(f)$ (deg-lex)

if $f \in \{0, 1\}$ **then**
  set $t := 1$
else if isCached(lead, f) **then**
  set $t :=$ cache(lead, f)
else
  set $x :=$ root variable of f
  if $\deg(f) = \deg(\mathrm{thenBranch}(f)) + 1$ **then**
    set $t := x \cdot \mathrm{lead}(\mathrm{thenBranch}(f))$
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Caching and Recursion

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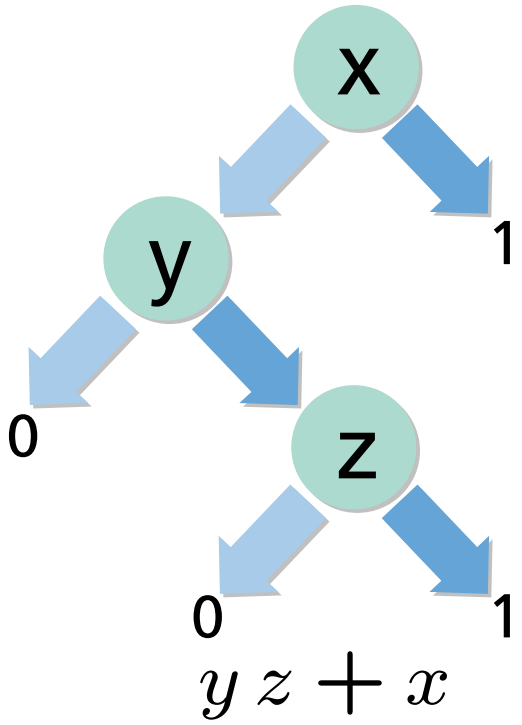

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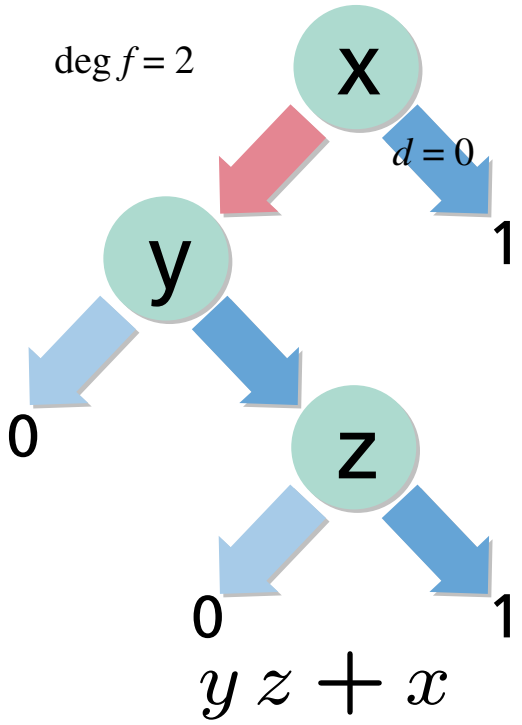

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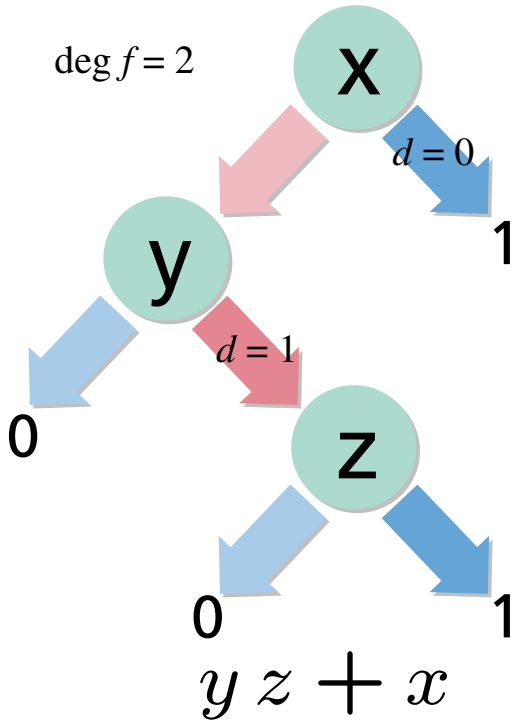

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Formal Verification

Boolean Polynomials and Logical Expressions

- Propositional logic $\longrightarrow$ Boolean polynomial
- Each formula from propositional logic can be stated using Or, Not, True, False
- Choose a mapping: True $\longrightarrow 0$, False $\longrightarrow 1$
  $\mathsf{Or}(x, y) \longrightarrow x \cdot y$, $\mathsf{Not}(x) \longrightarrow 1 + x$

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Property Checking

- Digital system given as set R of Boolean polynomials
- Property p (Boolean expression) to be verified w. r. t. R
- Check via normal form computation: $\mathrm{NF}(p, R) = 0$?

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Gröbner Basis

- Let $G = \{g_1, \ldots, g_m\}$ be (multivariate) polynomials
- Define an ordering $>$ on the monomials of $K[x_1, \ldots, x_m]$
- For a polynomial p define lead(p) the biggest monomial occurring in p

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G is a Gröbner basis $\iff$ for every $0 \neq p$ in the ideal generated by G there exists $g \in G$: lead(g) divides lead(p)

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Examples

- $\langle G \rangle$ is the whole ring, then $G \ni 1$
- $\{x_1 - c_1, \ldots, x_n - c_n\}$ for a (radical) ideal with unique solution $c_1, \ldots, c_n$

Linear Lead-Rewriting System

System of Boolean polynomials
- $G = \{g_1, \ldots, g_m\}$ in variables $x_1, \ldots, x_n$

Suppose, the following holds:
- For each i: lead(g_i) is a variable x_{gi} ($\rightarrow x_{gi}$ not in $g_i -$ lead(g_i))
- For $i \neq j$: lead$(g_i) \neq$ lead(g_j)

$\Rightarrow$ Then G is a Gröbner basis and
$R := G \cup \{x_i^2 + x_i | i = 1, \ldots, n\}$ is a Gröbner basis

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Example: $G = \{x - 1, y - 1\}$, $p = x \cdot y \cdot z \rightarrow \text{NF}(p, R) = z$

Special Situation in Formal Verification

- A digital circuit is described by a linear lead rewriting system
- Each output signal o_j is described by a Boolean functions $f(i_1, ...i_s)$ with respect to its input signals $i_1, \ldots, i_s$

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- $R := \{e_j | o_j \text{ output signal}\} \cup \{x_i^2 + x_i | i = 1, \ldots, n\}$
- For the system described by this equations, we would like to check a property p, which is given as Boolean expression
- Check via normal form computation: $\mathsf{NF}(p, R) = 0$?

Representation of Adder Blocks with Carry Bit

Variables

$s(0), \ldots, s(n-1)$ (sums), $c(0), \ldots, c(n-1)$ (carry bits),
$a(0), \ldots, a(n-1), b(0), \ldots, b(n-1)$ (inputs)

Topological Variable Order

$s(n-1), c(n-1), a(n-1), b(n-1),$
$s(n-2), c(n-2), a(n-2), b(n-2), \ldots$

Outputs before inputs, a, b reverse alternating (fast ZDD handling)

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Equations

$c(i) + carry_i$
$s(i) + a(i) + b(i) + carry_i$

with $carry_{-1} = 0$
and $carry_i = a(i) \cdot (b(i) + carry_{i-1}) + b(i) \cdot carry_{i-1}$

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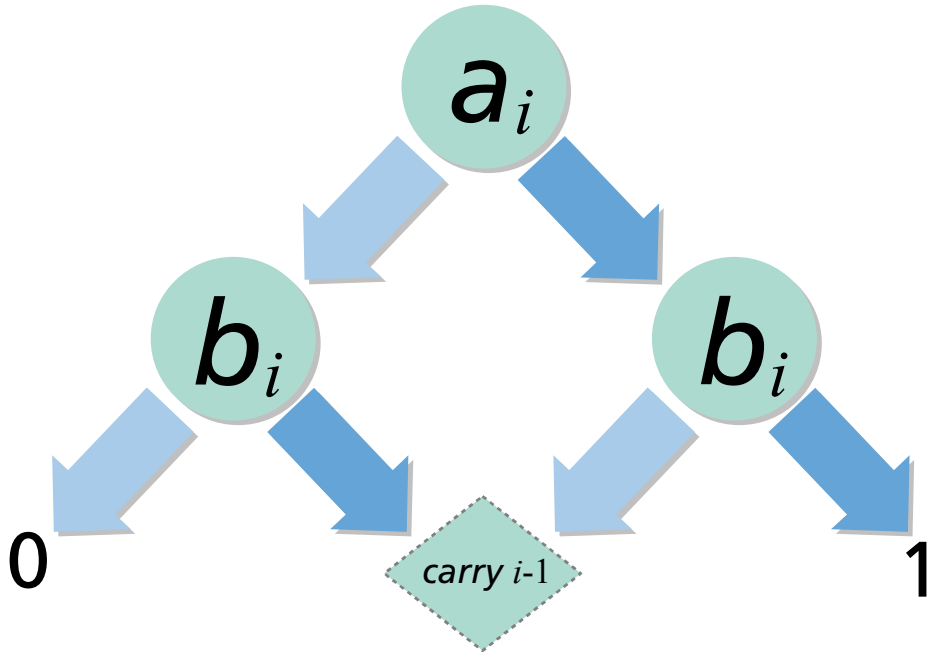
Carry-Bit Graph



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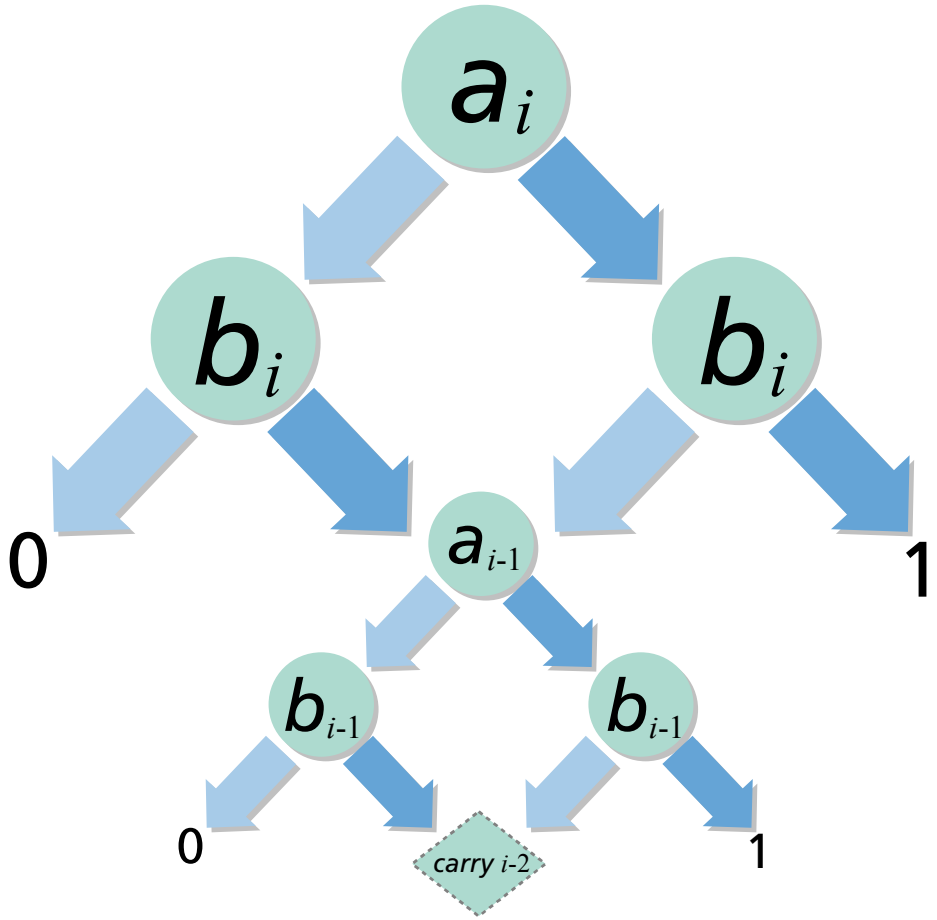

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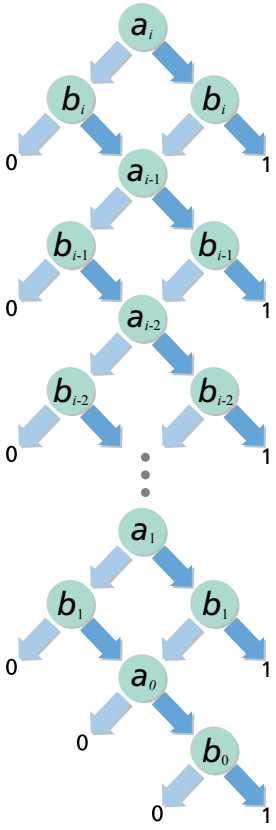

i-th bit	terms	nodes
1	1	2
2	3	5
3	7	8
4	15	11
5	31	14
6	63	17
7	127	20
8	255	23
9	511	26
10	1023	29
11	2047	32
n	$2^n - 1$	$3\,n - 1$

- Sophisticated ordering of variables
- Linear growth of ZDDs (plaited structure)
- Fast generation and arithmetic operations

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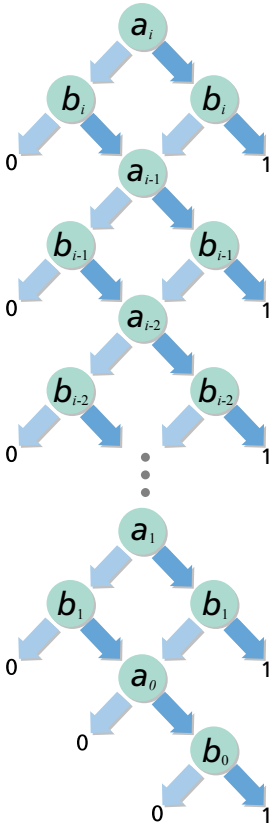

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Demo: $n = 4096$

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Application: Abstract 4096-Bit Adder and Logic

```
adder_bits=4096
adder_block=AdderBlock(sums="s",carries="c",
                       input1="a",input2="b",
                       adder_bits=adder_bits,start_index=1)

declare_ring([Block("x",100),adder_block,Block("y",100)])
ideal=[
    y(0)+y(1)+1,
    a(adder_bits)+1,b(adder_bits)+y(0)+y(1),
    x(0)+(c(adder_bits))*x(1)
]
adder_block.implement(ideal)

claims=[
  c(adder_bits)+1,
  if_then([x(1)],[x(0)]),
  if_then([a(1),b(1)],[c(1),s(1)]),
  if_then([a(1)+1],[c(1)+b(1),s(1)+b(1)+1]),
  s(adder_bits)+c(adder_bits-1),
  x(33)
]
```

Counter Example for Failed Claims

- A claim c fails $\iff$ $q := \mathsf{NF}(c, R) \neq 0$
- A Boolean polynomial is 0 $\iff$ it is 0 as a function

Algorithm

Boolean polynomial q reduced w. r. t. R, R red. Gröbner basis

$\mathsf{Variables}(q) \subseteq V := \{x_1, .., x_n\} \backslash \{\text{Leading Terms}(R)\}$

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Step 2 For every $x_i \in \mathsf{lead}(R)$, we have $p \in R$, with $x_i = \mathsf{lead}(x)$ and $\mathsf{tail}(p) \in K[V]$

Redefine $v_i := \mathsf{tail}(p)(v_1, \ldots, v_n)$

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Theorem

$v_1, \ldots, v_n$ provides a valid counter example

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Algorithm: Find a One of a Boolean Polynomial

Input: Boolean polynomial $p \neq 0$

Out: $\{x_i \to v_i\}_i : p(v_1, \ldots, v_k) = 1$

if $p = 1$ **then**
  return $\emptyset$ // empty tuple
end if
Set $r :=$ rootnode(p)
Set $x :=$ Variable corresponding to r
if elseBranch(r) is 0 **then**
  Set $v = 1$
  Set $q =$ thenBranch(r) // plug in 1
else
  Set $v = 0$
  Set $q =$ elseBranch(r) // plug in 0
end if
return $\{x \to v\} \cup$ find_one(q)

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Benchmarks: Pigeon Hole Problems

	Vars./Eqs.		PolyBoRi		MiniSat	
hole6	42	133	0.13 s	51.64 MB	0.01 s	1.96 MB
hole7	56	204	0.46 s	51.89 MB	0.06 s	1.96 MB
hole8	72	297	1.88 s	56.59 MB	0.30 s	2.08 MB
hole9	90	415	8.01 s	84.04 MB	2.31 s	2.35 MB
hole10	110	561	44.40 s	97.68 MB	25.20 s	3.24 MB
hole11	132	738	643.14 s	130.83 MB	782.65 s	7.19 MB
hole12	156	949	10264.92 s	338.66 MB	22920.20 s	17.13 MB

Remark

Good performance mainly due to fast Boolean multiplication

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Benchmarks: Lex. Normalform Computations without Gröbner basis

Interpolation vs. Gröbner approaches

# points	smallest lex. time/1 s	smallest lex. length	interpolate simple time/1 s	interpolate simple length	# basis
100	0.01	42	0.00	12771	287
500	0.06	249	$0.01 \approx 2.9 \cdot 10^{10}$		1943
1000	0.29	508	$0.01 \approx 8.1 \cdot 10^{13}$		3393
5000	5.53	2552	$1.47 \approx 4.5 \cdot 10^{23}$		10319
10000	19.78	5020	$37.18 \approx 1.6 \cdot 10^{26}$		17868
50000	250.95	25012			82929
100000	897.85	50093			162024
200000	3488.61	99868			296697
500000	20336.02	249675			636542

Remark

Normalform w. r. t. Boolean functions, given by points in $\mathbb{Z}_2^n$

Gröbner basis sizes calculated without explicit basis computation

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- Gröbner property checker for formal verification of given properties. Generation of counter examples, if claims fail
- Optimized setup of arithmetic components (adder)